VALIDATING VALUATION: HOW STATISTICAL LEARNING CAN CABIN EXPERT DISCRETION IN VALUATION DISPUTES

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This paper challenges conventional methods used in financial valuation across areas in litigation practice. We use a large-scale empirical simulation, using real firm data, to demonstrate that the widely used "comparable companies" approach allows enormous expert discretion, which enables substantial inconsistency and subjective judgment in court valuations. We then use the same simulation approach to show that using better data choices together with contemporary penalized regression methods generates valuation estimates that are considerably less variable, thereby reducing the scope for expert bias. We also apply this approach to a recent Delaware valuation dispute. This paper should transform financial valuation practice in litigation by both diagnosing and offering a cure for excessive discretion and variability in valuation disputes. Our methods would lead to better performing and more empirically grounded outcomes in legal disputes involving valuation, thus enhancing the fairness and efficiency of the judicial processes in valuation litigation.

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I. INTRODUCTION

For the past four decades, financial valuation has played an increasingly pivotal role in the litigation of high-stakes commercial disputes. While the judicial embrace of such methodologies was initially limited to isolated topics in corporate and securities law, the practice quickly expanded. Litigation has come to be influenced, and often dominated, by valuation disputes that hinge on financial economics—from bankruptcy to tax disputes, family law, fiduciary duties, and garden-variety questions in tort, property and contract law. By some accounts, the incursion of modern finance into commercial law has been nothing short of a pioneering revolution—a long-overdue hostile takeover of an "ossified, stagnant field".[1] Every top US law school now offers at least one course dedicated to teaching these techniques to law students.

However, the wholesale adoption of financial valuation in commercial litigation has had unfortunate collateral consequences. The first stems from the fact that financial economics tends to be a mathematical enterprise. Most judges are not formally trained asset-pricing specialists, even in business-intensive courts like Delaware's Court of Chancery; yet they are now typically required to admit, exclude, and sometimes weigh technical financial evidence, or even instruct lay juries possessing even less expertise. In most cases, judges are left to pick up key tenets of valuation practice on the fly, frequently (and understandably) relying on the motivated pedagogy of litigants or their hired experts.

Second, despite its seemingly precise technical facade, there is little doubt that financial valuation partakes of art at least as much as it does science. The field is awash with free parameters that require judgment and thus afford expert analysts considerable practical discretion. Experts can choose from an ample menu of general valuation approaches, each involving scope for experts to make judgment calls. Meanwhile, the professional literature meant to guide and substantiate financial experts' choices has itself grown varied; a variety of self-styled "authoritative manuals" contain distinct—sometimes inconsistent—formulations for best practices, further amplifying the need for judgment, and thus its close cousin, discretion.[2]

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[1] Roberta Romano. *After the Revolution in Corporate Law*. 55 J. L. Educ. 342, 342 (2005).

[2] *E.g.*, Ivo Welch, CORPORATE FINANCE ed. 5 (2023).

Third, as is customary in the American adversarial system, experts who deploy financial methodologies in the courtroom are typically compensated by one side's litigants. Many if not most are respected members of academic or professional finance circles, but they also tend to engage in repeat-play consulting work, which is largely shielded from public scrutiny. Although evidence law formally limits cherry-picking,[3] there are limits to the observability of what experts do with the data they use; some calculations may be left on the cutting room floor. Expert reports, moreover, may be filed under seal, keeping them out of public view until long after trial, perhaps indefinitely. Experts might thus embrace techniques in litigation that they would not in academic settings.[4]

The final problem arises because judges who adjudicate the claims of dueling financial experts generally must provide reasons for their judgments. The natural thing for a non-expert judge to do is adopt one or the other expert's assumptions or rationale for each specific issue. Sometimes the sum of such parts makes an incoherent whole. And even if not, once a judge's reasoning is memorialized in a written decision, it can become entrenched, serving as persuasive authority or even binding precedent that validates experts' discretionary choices.

In recent years, some courts have changed their practices as a result. For example, some have embraced the practice of routinely unsealing expert reports for public scrutiny, at least after trial, under the theory that the prospect of professional embarrassment can provide needed discipline.[5] Others have suggested structural reforms, such

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[3] *See, e.g.,* Federal Rule of Evidence 702(b) (requiring that expert testimony be "based on sufficient facts of data," which can be understood to prevent cherry-picking, *see, e.g.*, Jonah B. Gelbach, *Estimation Evidence*, 168 U. Pa. L. Rev. 549, 587 (2020)).

[4] *See e.g.,* In re of SWS Grp., Inc., No. CV 10554-VCG, 2017 WL 2334852 (Del. Ch. May 30, 2017); Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd, 177 A.3d 1, 36 (Del. 2017) ("As is common in appraisal proceedings,180 each party—petitioners and the Company—enlisted highly paid, well-credentialed experts to produce DCF valuations. But their valuations landed galaxies apart—diverging by approximately $28 billion, or 126%.").

[5] See Eric Talley, *Finance in the Courtroom: Appraising Its Growing Pains*, Delaware Lawyer (Summer 2017).

as having courts retain an independent expert to advise on valuation issues,[6] or experimenting with expert "hot-tubbing",[7] or committing to final-offer arbitration mechanisms to incentivize experts towards moderation.[8] Still others have instead attempted to skirt altogether the messy enterprise of valuation sausage making, by, for example, simply adverting to the negotiated merger price itself or pre-existing securities market prices.[9]

Although we believe that many of these institutional tweaks warrant consideration, this project follows a different path. We propose that courts adjudicating litigation in which valuation is at issue should require experts to demonstrate that the methods they use perform well at valuation. We argue that contemporary methods, especially so-called "comparable companies," can be viewed as a type of nearest-neighbor approach. We show that these approaches have numerous "expert degrees of freedom." For example: How many comparable companies will be included? How is comparability to be determined? How are comparable companies' values used to place a value on the company at issue in the litigation?. We show further that these expert degrees of freedom can be expected to lead to sizable gaps between opposing experts' valuations in litigation. Importantly, all of this is true even if experts

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[6] Andrew MacGregor Smith, Using Impartial Experts in Valuations: A Forum-Specific Approach, 35 Wm. & Mary L. Rev. 1241 (1994).

[7] *See* Dan Papscun, "Courtroom 'Hot Tub' Puts Google Trial Experts to Stress Test," Bloomberg Law, Oct. 6, 2023.

[8]*See* Keith Sharfman, Valuation Averaging: A New Procedure for Resolving Valuation Disputes, 88 MINN. L. REV. 357, 365-366 (2003).

[9] *See* Verition Partners Master Fund Ltd. v. Aruba Networks, Inc., No. CV 11448-VCL, 2018 WL 2315943, at *2 (Del. Ch. May 21, 2018) ("Under a traditional formulation of the efficient capital markets hypothesis, the unaffected market price provides a direct indication of the value of the subject company based on its operative reality independent of the merger, at least for a company that is widely traded and lacks a controlling stockholder.17 I therefore concluded on the facts presented that the most persuasive evidence of Aruba's fair value was its unaffected trading price."), *judgment entered sub nom.* Verition Multi-strategy Master Fund Ltd. v. Aruba Networks, Inc. (Del. Ch. 2018), and rev'd, 210 A.3d 128 (Del. 2019); Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd, 177 A.3d 1, 35 (Del. 2017) ("Taken as a whole, the market-based indicators of value—both Dell's stock price and deal price—have substantial probative value.")

use widely accepted methods; the discretion is created by the need for judgment about *how* to use such methods.

As we discuss in Part III, existing valuation methodologies are ultimately exercises in prediction. This is true whether they are used to estimate the value of a firm's equity, debt, enterprise value, or other target; valuation methodologies are worth deploying only if there is reason to think they will render a credible prediction of the target firm's value given the available data. As we explain, each of the three leading market valuation methodologies currently in use—comparable transactions analysis, comparable companies analysis, and discounted cash flow analysis—can be viewed as wholly or partly involving the logic of what is known in the machine learning literature as *k*-nearest neighbor (*k*-NN) prediction. Given that each approach aspires to deliver a prediction of a target firm's value on a target date, then, these methodologies differ only in the choice and use of data used to accomplish the task of prediction.

In Part IV.A, we use the *k*-NN framework together with Monte Carlo simulations using actual firms' data to demonstrate that the comparable companies approach to valuation both (a) entails enormous discretion, and (b) yields highly variable predictions. Both of these features are undesirable from an adjudication perspective, providing substantial reason to limit the use of conventional valuation methods.

In Part IV.B, we go further. We present much better ways to value firms than the comparable companies method (or ones that work similarly). We consider a variety of alternative approaches based on (i) using quarterly data on market capitalization data over the two years before valuation to predict a target firm's value on a target date; (ii) using *daily*, rather than quarterly, market cap data to predict target value; (iii) using daily data on *stock returns* to predict target value. In all three cases, we use penalized regression estimation methods. That's important, because such methods are highly data-driven, which reduces the scope for expert discretion. If expert degrees of freedom are the problem, then data-driven estimation likely is at least part of the solution.

Again using Monte Carlo simulations, we find that each of these approaches yields predicted valuations of target firms that are approximately centered on the target firm's true value on the target date. And critically, our simulations show that these methods are all far less variable than the comparable companies method. This is especially true for the approach that uses daily stock returns data, which entails much lower variation in target firm valuation than does any comparable companies method.[10]

Our project therefore makes two important contributions. First, using simulation evidence based on actual firm valuations, we show that current practices are highly variable and allow large expert degrees of freedom. Second, we show that market cap and/or stock returns data can usefully be used together with contemporary penalized regression techniques to leverage data-driven estimation; the result is both to reduce expert discretion and to improve the performance of valuation methods.

Our analysis proceeds as follows. In Part II, we provide an overview of standard valuation approaches that are prevalent in the literature, focusing on comparable transactions, comparable companies, and discounted cash flow analyses. Part III argues that comparable transactions and comparable companies methods can be viewed as *k*-NN prediction, with as-practiced discounted cash flow analysis sharing that feature at least partially. In Part IV.A, we spotlight comparable companies analysis and demonstrate the extent of expert degrees of freedom using a large-scale simulation of 10,000 target firms and dates. Then in Part IV.B, we propose and evaluate a series of competing approaches using market cap and stock price data together with data-driven prediction methods. In Part V we discuss Delaware's *DFC Global* valuation case as an illustrative example of how our proposed methods work in an actual case. We discuss limitations and extensions in Part VI, and then we conclude.

II. CONVENTIONAL PRACTICE

As it is currently practiced in business law courtrooms and boardrooms, modern valuation practice is dominated by three alternative methodologies: Comparable companies (CC), comparable transactions (CT), and discounted cash flow (DCF) analyses. In many cases, a valuation expert will attempt to value a financial asset of interest using two, or even all three of these

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[10] That finding is aesthetically satisfying as well, inasmuch as another main area in which financial economics is frequently used in litigation—securities fraud—regularly involves the use of daily stock returns.

approaches.[11] Particularly in cases of company valuations associated with merger agreements or bankruptcies, experts may use other valuation methodologies as complements. Such additional approaches include an analysis of historical premiums paid, analyst forecasts, and leveraged buyout/recapitalization analysis. When such alternatives are employed, however, they are typically offered only as "reference" valuations meant to complement CC, CT and DCF approaches.[12]

A. *Comparable Transactions*

The CT approach may be the most intuitively accessible, as it bears resemblance to the approach that real estate appraisers take when using "comps" to estimate the value of one's home. The basic idea is to find examples of analogous assets that have recently been sold in arm's length-transactions, and use those sales prices to deliver an estimate of what the sale of the company in question would deliver. With home appraisals, this process usually begins by selecting neighborhoods in a similar geographic area with similar traits, such as walkability, school quality, and income, as well as having a similar number of bedrooms and square footage. The recently sold properties deemed similar to the property in question along these dimensions are an appraiser's comps. The appraiser then will usually normalize the measure of value, such as price per square foot, for each comp transaction, and will aggregate the comps using the mean or median of the price-per-square-foot values. This yields a summary

[11] *See, e.g.* Highfields Cap., Ltd. v. AXA Fin., Inc., 939 A.2d 34, 46–47 (Del. Ch.), judgment entered, (Del. Ch. 2007) ("Shaked testified that the fair value of MONY as of July 8, 2004 was $43.03 per share. He reached this conclusion using three traditional valuation methodologies.")

[12] In re PetSmart, Inc., No. CV 10782-VCS, 2017 WL 2303599, at *24 (Del. Ch. May 26, 2017) ("Metrick asserts that his opinion regarding the fair value of PetSmart at the Merger Price is bolstered by the following confirmatory analyses: (1) his DCF analysis resulting in a value of $81.44 per share; (2) the fact that "[a]t no point prior to PetSmart's acquisition did its shares trade at or above $83 per share"; (3) the fact that "[a]t no point prior to the consummation of the transaction did analysts' average price target of PetSmart exceed $83 per share...").

measure of the price per square foot to be applied to the property whose valuation is in question.[13]

The CT approach for companies and other financial assets operates similarly. Much like the property appraiser, a valuation analyst using a CT methodology will first find recent sales of companies deemed comparable. If feasible, the analyst will limit attention to transactions in similar industries, geographic locations, or vintages.[14] Like real estate, companies vary in size, so finding comps of similar scale is desirable. In addition, they can have unique capital structure traits: for example, corporate debt often can transfer over as part of the sale, in which case the purchase price reflects only equity value. The valuation analyst will thus attempt to produce a measure of firm value that controls for both equity and debt factors. To control for capital structure, the analyst often needs to rescale the purchase price to reflect what is known as the "enterprise value" of each comparable company, adjusting the sales price of each comp to account for the value of any debt not capitalized into the sales price.[15]

Once comparable sales prices are converted to enterprise values, analysts then address the size factor, using an analog of the price per square foot measure used by real estate appraisers. Here, however, the standard normalized metric is typically an earnings multiple: That is, re-expressing the firm valuation as a multiple of some specified measure of earnings rather than in raw dollar terms. A standard metric that operates as a default is earnings before interest, taxation, depreciation and amortization (EBITDA), which is often a

[13] Either a point estimate or a range of estimates might be provided.

[14] These and other traits are specified in a small set of valuation manuals that have become accepted over time in the profession.

[15] *See, e.g.* In re Appraisal of Jarden Corp., No. CV 12456-VCS, 2019 WL 3244085, at *49 (Del. Ch. July 19, 2019), on *reargument in part sub nom.* In re Jarden Corp., No. CV 12456-VCS, 2019 WL 4464636 (Del. Ch. Sept. 16, 2019), *and aff'd sub nom.* Fir Tree Value Master Fund, LP v. Jarden Corp., 236 A.3d 313 (Del. 2020) ("In order to determine the final share price under a DCF approach, the appraiser must account for Jarden's excess cash and debt in its enterprise value.") Other adjustments include netting off cash (and cash equivalents), as well as making sometimes- controversial changes to working capital. *See, e.g.,* OSI Sys., Inc. v. Instrumentarium Corp., 892 A.2d 1086 (Del. Ch. 2006).

relatively stable proxy for cash flows in mature companies.[16] For less mature companies, however, it is not uncommon to see multiples based on other factors, such as EBIT, sales revenues, or, less commonly, other measures of market interest such as "clicks" on the company's website.[17] Non-EBITDA multiples are typically disfavored,[18] however, and tend only to be used when the company generates negative adjusted earnings, which renders any multiples-based approach nonsensical.

Even after a multiple has been selected, there are many ways to quantify the denominator of the multiple. For example, one might base a multiple on the last fiscal year's numbers, or the last twelve months, or projections for the next twelve months or fiscal year. In each case, the multiple of the comparable company may change, and it is not uncommon for analysts to assess CT multiples using several measures, thereby cobbling together a range of valuations based on the aggregated outcomes of such approaches, as well as a variety of permutations of the mean, median or inter- quartile range for each measure. In formulating the comps, the multiple formulations, and the

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[16] Doft & Co. v. Travelocity.com Inc., No. CIV.A. 19734, 2004 WL 1152338, at *10 (Del. Ch. May 20, 2004) ("Gompers agrees with Purcell that the EBITDA multiples are the 'preferred multiple to examine' because they 'are closest to cash flow and are a better proxy for the firm's on-going concern value.'").

[17] Merion Cap., L.P. v. 3M Cogent, Inc., No. CV 6247-VCP, 2013 WL 3793896, at *6 (Del. Ch. July 8, 2013), *judgment entered sub nom.* Merion Cap., L.P v. 3M Cogent, Inc. (Del. Ch. 2013) ("The comparable companies method of valuing a company's equity involves several steps including: (1) finding comparable, publicly traded companies that have reviewable financial information; (2) calculating the ratio between the trading price of the stocks of each of those companies and some recognized measure reflecting their income such as revenue, EBIT, or EBITDA; (3) correcting these derived ratios to account for differences, such as in capital structure, between the public companies and the target company being valued; and, finally, (4) applying the average multiple of the comparable companies to the relevant income measurement of the target company, here Cogent.")

[18] Our simulation analysis focused on an EBIT-based measure for data availability reasons. We note that this is not a rare occurrence; in our data sample for the simulation, nearly 30% of firms have negative EBITDA values. Thus, we focus on EBIT in scaling earnings to increase sample size in our analyses.

ranges, the analyst typically retains significant discretion—an issue to which we return in our analysis below.

However, two potential constraints on the CT approach often limit its ability to deliver valuation projections. The first is a lack of data. Because bona fide arms-length sales of companies within a given industry are generally rare, the set of "comparable transactions" might itself be relatively sparse, which may force the analyst to make a projection from a very small group (perhaps even as small as one). One potential solution to this problem is to lengthen the time horizon or broaden the criteria by which comparatives are drawn (e.g., by expanding the industries considered). The second potential limitation on the CT approach is that it is predicated on sales of comparable firms through negotiated transactions. Such sales often come with a premium baked into the sales price, reflecting the value of control. This baked-in control premium may sometimes be inappropriate if (for example) one is interested in gauging only the cash flow valuation of the company. In such settings, CT requires an attempt to shave off control premia from precedent sales.

B. *Comparable Companies*

The CC approach is a close cousin to the CT approach, differing only in the source of the data used to assess comparable firms. While CT uses sales price data from acquisitions of comparable firms, CC uses the full spectrum of data from large and thick securities markets. In thick markets, stock prices are thought to be a good proxy for the economic value of a fractional share of the company, at least on average and as viewed by the marginal investor.[19] Thus, rather than using the sales price to predict value, the total market capitalization of comparable firms can be based on public trading data. Beyond that, however much of the CC valuation process is identical to that in CT, including the conversion from an enterprise value multiple to enterprise value, the specification of the multiple itself, the use of judgment about how to measure such multiples, such as last twelve months, next twelve months, etc., and a summary measure (mean or median) for aggregating comp multiples. In other words,

[19] Verition Partners Master Fund Ltd. v. Aruba Networks, Inc., 210 A.3d 128, 137 (Del. 2019) ("Dell's references to market efficiency focused on informational efficiency—the idea that markets quickly reflect publicly available information and can be a proxy for fair value."

beyond the different source of valuation metrics for the comparable firms, nearly every other part of a CC analysis tracks the CT analysis almost directly.

CC methodologies have one obvious advantage over CT approaches: data. It can be difficult to find precedent M&A transactions to use in developing CT comps, but CC is facilitated because thousands of public companies trade continuously and have observable prices each day. Consequently, CC allows one to build sizable groups of comparable firms. On the other hand, CC also tethers the value of companies to the trading value of their stocks, which in turn tends to reflect the value the market ascribes to a publicly held valuation target. Thus, the CC approach might neglect the value of the control premium. Another potential limitation of the CC approach is that it depends critically on the on-average value efficiency of trading markets. This may often be appropriate, but CC fits less well when the target firm is traded in a thin, volatile, poorly-developed market where market valuations and fundamental valuations can diverge.

C. *Discounted Cash Flow*

The third major form of valuation is discounted cash flow (DCF) analysis. Compared to CC and CT approaches, the DCF has significantly more moving parts, and is generally viewed as more technically demanding.[20] Rather than looking for comparable firms (at least directly), the DCF approach conceives of the value of a financial asset as the equivalent to the present discounted value of the free cash flows the asset is projected to produce. Borrowing from the

[20] *See, e.g.,* Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd, 177 A.3d 1, 37–38 (Del. 2017) ("Although widely considered the best tool for valuing companies when there is no credible market information and no market check, DCF valuations involve many inputs—all subject to disagreement by well-compensated and highly credentialed experts—and even slight differences in these inputs can produce large valuation gaps. Here, management's projections alone involved more than 1,100 inputs, and the experts' fair value determinations (which also included several novel tax issues discussed below) landed on different planets."

well-known formula in finance for present values, the DCF approach[21] can be captured as follows:

$$FMV = PV(Cash\ Flows) = \frac{FCF_1}{(1+WACC)} + \frac{FCF_2}{(1+WACC)^2} + \cdots + \frac{FCF_T}{(1+WACC)^T} + \frac{S_T}{(1+WACC)^T}$$

where $t = 1, 2, 3, \ldots T$ indexes time period, with period T being the terminal period; FCF_t denotes expected future "free cash flows" in period t; S_t denotes a terminal (or "salvage") valuation of the asset as of the terminal projection year; and WACC represents a risk-adjusted discount rate known as the "Weighted Average Cost of Capital." If all of these ingredients are known, or can be reliably estimated under different scenarios, then a cash-flow prediction is possible.

Like an onion, each of the ingredients of the DCF approach has its own layers of complexity (and resulting expert discretion). Free cash flows are sometimes generated from analyst forecasts, or through management forecasts done in the ordinary course, or through investment banker forecasts for the purposes of a (disputed) transaction, or by some other source. They are typically, though not always, unlevered, and thus do not carve off interest payable to capital creditors, so as to summarize the entire pool of earnings available to satisfy both debt holders and equity holders. And, in some cases cash flow projections of comparable firms (if available) can be used to form composite projections that more closely track the industry at large.

The WACC discount rate is typically a blend of expected return estimates for debt and equity (adjusted for leverage ratios and tax deductions), with equity return estimates the product of an underlying asset pricing model (such as the still-dominant market "beta" from the capital asset pricing model). In many cases, peer company betas are also blended in with company-specific data to create more of

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[21] *See, e.g.* In re Vanderveer Ests. Holding, LLC, 293 B.R. 560, 578 (Bankr. E.D.N.Y. 2003).

a composite measure (usually after an elaborate process adjusting peers' betas for differences in peer leverage ratios).[22]

Finally, the terminal value measure (S_T) represents something of a capitulation to our inability to make projections indefinitely into the future. Because company projections are typically no longer than 10 years (and are more frequently 5-7 years), an analyst using DCF must make an assumption of what the asset will be worth in its terminal period (when no more forward-looking projections are available). The assessment of terminal value is perhaps the most susceptible to expert degrees of freedom, since there is little to tether it to firm-level data. One approach for terminal values is to extrapolate the final period's free cash flow indefinitely into the future as a "growing perpetuity" at some posited growth rate, backing out a present valuation (as of period *T*) from a well-known formula for the present value of growing perpetuities.[23] Another frequently used approach, however, is simply to revert (once again) to peer company multiplies using a recycled CC or CT approach applied as of the terminal period.[24]

III. The Conventional Approach as *K*-Nearest Neighbors Matching

In Section 2 we described conventional approaches to valuing firms for litigation purposes. Experts retain substantial discretion in how they go about delivering a valuation—discretion that ranges from the choice of approach to how they measure the inputs of their valuation metric. Second, peer-group comparisons are pervasive—indeed such comparisons are the very backbone of CC and CT analysis, and even in DCF analysis, peer group comparisons sneak in in myriad ways (free cash flow projections, terminal values, beta estimates, etc.).

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[22] *See, e.g.* Ramcell, Inc. v. Alltel Corp., No. 2019-0601-PAF, 2022 WL 16549259, at *19 (Del. Ch. Oct. 31, 2022), judgment entered, (Del. Ch. 2022).

[23] *See, e.g.* In re Vanderveer Ests. Holding, LLC, 293 B.R. 560, 578 (Bankr. E.D.N.Y. 2003).

[24] *See, e.g.* Ramcell, Inc. v. Alltel Corp., No. 2019-0601-PAF, 2022 WL 16549259, at *19 (Del. Ch. Oct. 31, 2022), judgment entered, (Del. Ch. 2022).

When considering the scope of the expert's task, it is noteworthy how closely valuation practice resembles the *k*-nearest neighbor (*k*-NN) algorithm, both generally and in specific application. First, valuation experts' goal is to deliver core predictions, rather than to test causal theories; (2) valuation experts use a significant amount of unstructured data to deliver those predictions; and (3) valuation experts habitually use claimed similarities of instances—styled as comparable firms, or "comps," in their predictive enterprise.

The *k*-NN algorithm, first developed over a half century ago,[25] is a non-parametric method for statistical learning that's often used to classify or predict a target variable of interest. The parameter *k* is the number of closest neighbors of the target that are used to create the prediction. In our context, the target variable is some measure of firm value for a target firm at a particular moment. When multiple features of firms are available for use in value prediction, the set of neighbors that are closest to the target will depend on how experts choose to map these multiple features into a measure of distance between firms, as we discuss in more detail below. The *k*-NN method can be used for both classification problems, where the object is to predict which of a discrete set of categories the item of interest belongs, or continuous problems, where the objective is to predict the value of a continuous outcome variable, such as our present context, in which firm value is the target.

In the simplest *k*-NN setting, only one variable, x, is used in matching, and the *k*-nearest neighbors are the *k* firms with the closest values of x to the target unit. These *k* firms function as valuation experts' comps. The predicted firm value for the target firm is then the average of the *k* comps' values (the median of the comps' value is also sometimes used). The *k*-NN algorithm can be generalized to allow more than one *x*-variable. Speaking generally, its key ingredients are

[25] See, e.g., Evelyn Fox & Joseph Hodges, Discriminatory Analysis. Nonparametric Discrimination: Consistency Properties vol. 4 USAF School of Aviation Medicine 1951 and Thomas Cover & Peter Hart, Nearest Neighbor Pattern Classification, 13 IEEE Trans. Info. Theory 21 (1967).

(a) an outcome variable y, whose value (often called a "label" in machine learning communities) is to be predicted;

(b) a vector of characteristics, X, whose values will be used to assess the distance between the target and its neighbors;

(c) a distance metric, which is a function that, when applied to any pair of X-vector values, measures the distance between them (one example of a distance metric is Euclidean distance);

(d) a rule for selecting which neighbors will be used to predict the target unit's value of y (*e.g.,* "use the 3 closest neighbors to the target firm based on the distance metric in step (c)"); and

(e) a function that combines the values of the selected neighbors' values, $\{y_i\}_{i=1}^{k}$, into a single prediction, $\hat{y}$ (*e.g.*, the mean, median, or inter-quartile range of the selected neighbors' values).

This discussion reveals that the Comparable Companies and Comparable Transactions valuation methods are not just *similar* to the *k*-NN algorithm—they are *instances* of the *k*-NN algorithm. Both Comparables approaches involve

(a) using an earnings multiple as the outcome variable y (for CC, y is trading value, and for CT it is acquisition value);

(b) using a set of firm variables as the X data used to identify comparable companies or transactions;

(c) & (d) deciding how comparable peer firms are to the target firm and which of these comps should be used to predict the target firm's earnings multiple (in practice these steps are rarely specified precisely); and

(e) using some aggregation method—such as the mean or the median—to predict the target firm's earnings multiple.

To the best of our knowledge, neither the CC method nor the CT method has previously been viewed as an instance of the *k*-NN algorithm, but it is clear from the discussion above that this identification is accurate.

The DCF valuation method does not map so neatly into the *k*-NN algorithm. But it still shares some important characteristics. For example, it is common in DCF practice to use comps to generate estimates of terminal value, of earnings projections, and of asset betas—all three of which are core ingredients of DCF valuation methods. Thus, *k*-NN's spirit, if not always its letter, may enter DCF analysis at several junctures.

Given that standard valuation methodologies are, or are similar to, *k*-NN learners, we can understand their pros and cons by reference to *k*-NN's. The *k*-NN method is a form of instance-based learning, which means that, unlike other supervised learning approaches, it does not require a training stage for use.[26] That makes k-NN simpler and faster to use than other algorithms. And when the data set grows large, k-NN regressions are known to be Bayes optimal.[27]

On the other hand, *k*-NN has several limitations that can hamper its performance. First, as the amount of data grows, the cost of calculating distances increases very rapidly, because pairwise distances must be calculated.[28] *k*-NN use is also complicated by the potentially large number of X variables,, which can lead to the so-called "curse of dimensionality".[29] Third, *k*-NN can be highly sensitive to scale and how distance is calculated. Good performance using *k*-NN often requires normalization of variables before applying the algorithm—which adds a layer of expert discretion, because multiple normalizing scalers exist. Finally, *k*-NN can be sensitive to noisy data, missing values, and outliers.

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[26] David W. Aha, Dennis Kibler, and Marc K. Albert. *Instance-Based Learning Algorithms.* 6 Machine Learning (1991).

[27] [Say something about why that's good]___

[28] Trevor Hastie, Robert Tibshirani, Jerome H. Friedman, and Jerome H. Friedman. *The Elements of Statistical Learning: Data Mining, Inference, and Prediction.* 480 (2d Edition, 2009). This is unlikely to be a serious problem in firm valuation problems, so long as the set of potential comps can be limited on preliminary grounds, e.g., by focusing on firms that share common industries of operation.

[29] Gareth James, Daniela Witten, Trevor Hastie, and Robert Tibshirani 106-107, 161 *An Introduction to Statistical Learning* (2d Edition, 2013).

This discussion indicates that actual *k*-NN performance will depend on a number of factors whose net contributions are hard to characterize analytically. Accordingly, we turn next to a wide-ranging collection of Monte Carlo simulations to assess the performance of *k*-NN, as well as other data-driven approaches, using real firm data.

IV. MONTE CARLO SIMULATIONS USING *k*-NN

In this section, we conduct Monte Carlo simulations using actual firm data to assess the performance of *k*-NN and alternative data-driven valuation approaches. We draw our data primarily from Compustat (financial reporting information) and the Center for Research in Security Prices (CRSP) (stock price data). We use quarterly financial data from 2000 to 2020 for all public reporting companies in the United States with a fiscal-year reporting date of December 31.[30] We merge this data with CRSP stock price data using the historical linking file provided by Wharton Research Data Services (WRDS). We keep only confirmed links,[31] and we match to daily individual security prices and index returns from the CRSP daily stock and index files, as well as the daily factor returns from Ken French's website (i.e. the "Fama-French-Carhart Factors").[32]

We next create a series of covariates for choosing peers using based on leading the discussion in two leading corporate

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[30] We restrict our attention to 12/31-reporting firms so that the different fiscal-year quarter ends align in calendar time. In additional results that are available on request, we find that this restriction does not drive our results.

[31] These are link codes equal to "LC", "LU", or "LS".

[32] This file is available at the url https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

finance textbooks.[33] We drop any observation on a firm for which any of the covariates we use is missing.[34]

In addition, we categorize each firm-quarter observation as belonging to an industry, defined by the first two digits of the firm's Standard Industrial Classification (SIC) code. We use the historical identifier, *sich*, which allows for time variation in industry designation.[35] We impute missing entries of *sich* using a "down-up" strategy, which assumes (i) that all missing values before the first recorded non-missing entry are equal to the first entry, and (ii) that subsequent missing values are equal to the most recent entry until a new entry is recorded.[36] We drop all observations for firms in the financial services industry (SIC code beginning with 6), because their capital structure and investment policies are significantly different from those in other industries.

Having constructed a panel dataset of firm-quarter observations, we then randomly sample 10,000 target firm-quarter observations for use in our simulations. In order to be selected as a target observation, we require firm-quarter observations to have (i) at least nine peer firms that have full non-missing data in the selected quarter, and (ii) at least eight consecutive non-missing values, ending on the selected quarter, for the ratio of market capitalization to EBIT (earnings before interest and taxes). In addition, for each target observation, we choose a simulated valuation date by

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[33] We provide a list of firm-quarter variables, and their calculation method and data source in the Data Appendix.

[34] To be clear, we do not necessarily drop *firms* in this case, but rather just the observations that have missing values; for example, in our daily analysis below, if firm *f* has T_f daily observations with non-missing data and M_f daily observations with some missing data, we would include the T_f observations (provided that they meet any other selection criteria).

[35] For our purposes, it is important to use *sich* rather than the commonly used *sic* variable, because the value of the latter is the most recently identified industry code for all observations. Consider a firm that shifts from industry A to industry B at time $t-1$. If Compustat data is downloaded at time *t*, the value of the *sic* variable will be industry B's code for all time periods (even though the firm switched industries).

[36] If *sich* has a missing value for the last *S* observations, we use the value of *sic*.

selecting a random number of trading days, d^*, between zero and fifty, after the end of the quarter to value the firm. For example, if the randomly selected observation is for the third fiscal quarter of 2015, which ends on September 30, 2015, and we draw $d^* = 45$, then the target valuation date is December 3, 2015, because this is the 45th trading date following third-quarter's end. We refer to the 10,000 firms and valuation dates that result from the process described in this paragraph as firm-date targets.

For each of our 10,000 firm-date targets, we know the following:

- The true *firm valuation* (understood as market capitalization).
- The true *valuation ratio* (the ratio of market capitalization to EBIT)
- Lagged values of the firm valuation and valuation ratio (*i.e.*, values for the period before the valuation target date).
- An industry categorization.
- A set of covariate values.

Our simulation entails using various prediction methods for each of the 10,000 firm-date targets, which yields a prediction value for the firm valuation and valuation ratio target variables for each combination of prediction method and firm-date target. We then calculate the difference between the *predicted* and the *true* values of the target variables for each prediction method and firm-target date. Finally, we assess the performance of each method by comparing statistics related to these differences.

To guide our discussion, we will use a (randomly-sampled) firm-target date for demonstrative purposes in the sections below. Table 1 shows that the firm is Landstar System, Inc. The target valuation quarter is the 4th quarter of 2017; with $d^* = 43$, the target valuation date is the 43rd trading day of 2018, which was March 5, 2018.[37] The firm is in industry code 42, representing motor-freight

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[37] *See, e.g.,* the tab for 2018 at the website https://www.swingtradesystems.com/trading-days-calendars.html.

transportation.[38] Its market capitalization was $4.371 billion, and its EBIT was $70, for a valuation ratio of 62.40.

Table 1: Motivating Example

Company	Quarter	Industry	Market Cap (M)	EBIT	Ratio	d^*
LANDSTAR SYSTEM INC	2017Q4	42	$4,371	$70	62.40	43

A. *Conventional k-NN Approach*

In Section 3 we described how the conventional approach to valuation described in leading corporate finance textbooks represents an instance of k-NN matching. In this section we use our simulation approach to test the performance, and susceptibility to expert discretion, of this conventional approach.[39]

Our Monte Carlo simulation proceeds as follows. First, for target firm j and target quarter q, denote the target-quarter valuation ratio r_{jq}.[40] Denote the firm's market capitalization on the target date d^* as mc_{jd^*}, and denote its valuation ratio on that date as r_{id^*}.[41] Our

[38] See, e.g., https://www.dnb.com/content/dam/english/economic-and-industry-insight/sic_2_digit_codes.xls.

[39] To be clear, we are not arguing that our k-NN implementation exactly replicates how experts produce valuation estimates under the conventional approach. It should instead be seen as a systematic implementation of the textbook approach that leaves open numerous areas of discretion. By clarifying the design considerations in the matching algorithm, our k-NN approach is, if anything, likely to have less discretion than implementing the CC method as the latter is used in litigation. Inasmuch as our point is to demonstrate that there is a large amount of discretion, our approach is thus methodologically conservative.

[40] This is the ratio of firm j's end-of-quarter market cap, mc_{jq}, to its end-of-quarter EBIT, $EBIT_{jq}$, so that $r_{jq} = \frac{mc_{jq}}{EBIT_{jq}}$.

[41] Thus, $r_{jd^*} = \frac{mc_{jd^*}}{EBIT_{jq}}$; notice that we use the daily value of market cap for date d^*, together with the value of EBIT for the preceding quarter, q.

prediction exercises below will target these four market cap and valuation ratio variables.

Having defined these target variables, we next identify all viable peer firms for a target firm and quarter. For each firm, we limit consideration to other firms (i) that are in the same two-digit SIC code industry, (ii) that have non-missing entries for X covariates, and (iii) have full market trading data for the 250 trading days prior to quarter end, and the d^* days following the quarter. Table 2 lists the potential peer firms that satisfy these three criteria for our motivating example, Landstar System with target quarter 2017Q4 and $d^* = 43$.

Table 2: Potential Peer Firms Within Same Industry

HUNT J B TRANSPORT SERVICES INC	FORWARD AIR CORP
WERNER ENTERPRISES INC	KNIGHT SWIFT TRANSPORT HDLGS INC
P A M TRANSPORTATION SVCS INC	COVENANT TRANSPORTATION GRP INC
MARTEN TRANSPORT LTD	CH ROBINSON WORLDWIDE INC
HEARTLAND EXPRESS INC	SAIA INC
OLD DOMINION FREIGHT LINE INC	UNIVERSAL LOGISTICS HOLDINGS INC

This table reports the potential peer firms for our motivating example. Landstar System Inc. had a two-digit SIC code of 42 in Q4 2017, which is Motor Freight Transportation and Warehousing. The 12 peer firms in this table are those with the requisite data over the time period.

Next, we calculate predictions for each target firm-date using k-NN. As we've discussed, there are multiple discretion-according parameterizations involved. Here we focus on four: (i) which X variables to use to assess firms' distances from each other; (ii) how many neighbors to use (i.e., the value of k); (iii) how to measure distance; and (iv) whether to aggregate comp firms' valuation variables using the mean or the median.

We now illustrate how using our motivating example. Table 3 lists the five nearest and farthest potential Landstar peers (among those that were listed in Table 2) using the matching variables specified in the Rosenbaum & Pearl corporate finance textbook. The distance between Landstar System and each potential peer firm was calculated using the scaled Euclidean distance metric. If we were to use k=5, then our comps would be Hunt (JB) Transport Services, Werner Enterprises, P.A.M. Transportation Services, Martin Transport, and Heartland Express. Our final step would be to aggregate the values of these firms. If our target variable is the ratio of market cap to EBIT, r_{jd^*}, then we would use 77.55, as this is the median value of the Ratio column in Table 3 for the five comps. That is, $\hat{r}_{jd^*} = 77.55$. This is the predicted ratio of market cap on date d^* to EBIT at the end of the most recent quarter before d^*. Thus, the comps-based predicted value of Landstar System's market cap on date d^* may be found as the product of $\hat{r}_{jd^*}$ and $EBIT_{jq}$. The EBIT value for Landstar Systems at the end of 2017Q4 was \$70 million (*see* Table 1). Therefore our predicted market cap for Landstar System is $\widehat{mc}_{jd^*} = \hat{r}_{jd^*} \times EBIT_{jq} = 77.55 \times \$70M = 5{,}429M$, or roughly \$5.4 billion.

Table 3: Motivating Example—Best and Worst Nearest Neighbor Matches

Company	EBIT	Market Cap (M)	Ratio	Rank
Best Matches				
HUNT (JB) TRANSPRT SVCS INC	166.17	\$12,886	77.55	1
WERNER ENTERPRISES INC	44.29	\$2,706	61.09	2
P.A.M. TRANSPORTATION SVCS	3.03	\$225	74.12	3
MARTEN TRANSPORT LTD	13.72	\$1,187	86.53	4
HEARTLAND EXPRESS INC	3.04	\$1,625	534.68	5
Worst Matches				
KNIGHT-SWIFT TRPTN HLDGS INC	137.40	\$8,511	61.94	8
COVENANT LOGISTICS GROUP INC	15.65	\$413	26.42	9
C H ROBINSON WORLDWIDE INC	210.88	\$12,600	59.75	10
SAIA INC	23.22	\$1,810	77.97	11
UNIVERSAL LOGISTICS HLDGS	13.11	\$607	46.32	12

This table reports the five best and five worst matches for our motivating example using a *k*-nearest neighbors matching approach. We use the covariates from Rosenbaum and Pearl (RP), and a scaled Euclidean distance metric.

Even given that we targeted the valuation ratio, there were four dimensions of discretion in the approach just described: the set of matching variables (Table 3 uses the Rosenbaum & Pearl set), the number of peers used as comps (our discussion of Table 3 uses 5), the distance metric (Table 3 uses the scaled Euclidean metric), and the aggregation measure (our discussion uses the median). In practice, an expert has discretion over these choices and can make them in whatever way benefits the side the expert favors.[42] To explore the scope of this discretion, we predict firm value using 24 different combinations of k-NN parameterization choices, with each combination found by combining choices from among each of the following bullet points:

- Two covariate sets (those described in Rosenbaum & Pearl (RP), or those described in Pratt & Niculita (PN)).[43]
- Three values of k (5, 7, or 9).
- Two distance metrics (scaled Euclidean (SE) or Mahalanobis (M)).
- Two ways of aggregating comps' valuation ratios (mean or median).

For each of the 24 combinations, we calculate a predicted valuation ratio, $\hat{r}_{jd^*}$, for firm j and date d^*, just as we did in discussing the Landstar System example in Table 3 above. We then multiply that predicted value by the actual $EBIT_{jq}$ for target firm j and quarter q. The resulting product is our market cap prediction, $\widehat{mc}_{jd^*}$, for the particular combination. There are 24 such values for each firm.

Figure 1 reports these predicted valuations for Landstar System. The top panel displays the estimates in order from smallest to largest. The range is wide: by choosing among the 24 options, experts could generate estimates as low as \$3.8 billion, and as high as \$9.7 billion. The true valuation on date d^*, represented by the dashed line in the top panel, was \$4.5 billion.

The figure's bottom panel displays the specific parameterization for each estimates, with shaded gray cells indicating the values of the four inputs used. For example, the lowest predicted value (\$3.8 billion) occurs when the RP covariate

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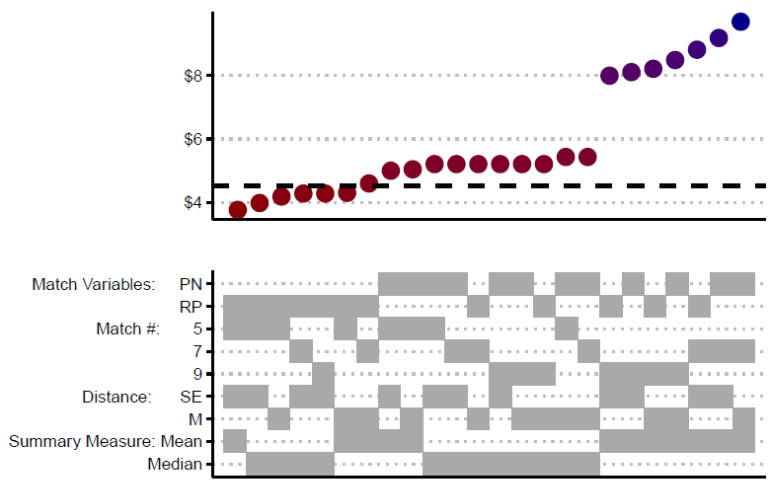
[42] Alternatively, a litigant could shop for multiple experts and submit to the court only the most favorable report. For discussions of such "expert mining," *see* Jonah B. Gelbach, *Expert Mining and Required Disclosure*, 81 U. Chi. L. Rev. 131 (2014); Christopher Tarver Robertson, *Blind Expertise*, 85 N.Y.U. L. Rev. 174 (2010).

[43] A list of these matching variables and their construction is presented in the Data Appendix.

set is used, $k = 5$ peer firms are used as comps, the scaled Euclidean distance metric is used, and comps' valuation ratios are aggregated using the mean. Perusing the bottom panel, we see that there is no obvious regularity linking parameterization choice to predicted valuation magnitude; each value of each of the four inputs is associated with both high and low predicted market cap values.

Figure 1: Input Choice and Valuation



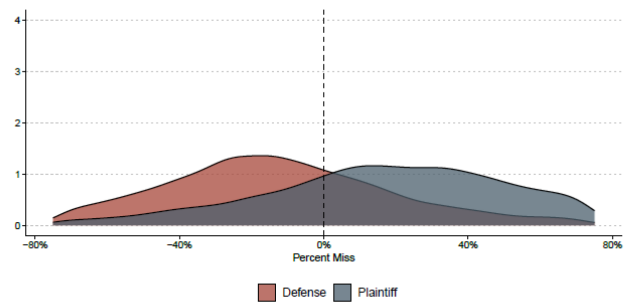
This figure reports the valuation estimates for Landstar $d^* = 43$ trading days after the end of 2017 Q4 using different input combinations. The top panel plots the estimated market value from each of the 24 combinations, which are represented by the grey tiles in the lower panel. The combinations vary based on the choice of matching feature set, number of matches, distance measure, and summary measure.

Figure 1 thus illustrates the great degree of expert discretion that conventional valuation approaches can allow. We believe this fact helps explain why expert valuation reports in litigation typically vary so widely between defense and plaintiff experts: Even restricting consideration to approaches that are squarely within textbook accounts of best practice, we find an enormous range of variation.

Of course, that analysis was for just one firm-date target. Perhaps there is something especially noisy going on for Landstar System in the 4th quarter of 2017. This is why we calculate the same

24 predicted valuations for each of our 10,000 randomly-selected firm-date targets. For each target, we calculate the predicted valuations for each of the 24 input permutations (two choices of matching variables, three choices of matching number *k*, two different distance measures, and two summary measures). We then assume that a defense expert picks the second-lowest estimate for their report and that the plaintiff chooses the second highest.[44]

Figure 2: Distributions of Hypothetical Defense-Expert and Plaintiff-Expert Predicted Valuations



This figure reports the distribution of the second-lowest and second-highest valuation estimates from the permutations of the conventional k-NN approach, measured by the percentage deviation from the true realized market capitalization on the trading date d∗ periods after the end of the fiscal quarter. We assume that the defense expert picks the second lowest valuation, and the plaintiff expert selects the second highest.

[44] An alternative way to interpret the strategy would be to assume that the litigation team surveys experts and selects the second lowest or highest estimate. Given the cost of producing expert analyses, this seems like the less plausible thought experiment.

Figure 2 plots kernel density estimates for the percentage by which these hypothetical defense and plaintiff valuations miss the true firm value.[45] To keep outliers from distorting this picture, we restrict it to estimates that are within 75% of the true value. The estimates separate into two visually distinct distributions having modes roughly at 25% (hypothetical plaintiffs) and -25% (hypothetical defendants). This yields the striking conclusion that the expert degrees of freedom even *within* *k*-NN methodology is roughly half the value of the target firm. This suggests that the common criticism of valuation disparities by the judiciary[46] could simply be driven by experts exploiting the degrees of freedom afforded by the conventional k-NN approach.

One potential remedy to this problem would be to use an aggregate measure of the 24 predicted valuations. We do this in Figure 3. In Panel A, the gray-shaded curve is the kernel density for the distribution of the medians over the target firm-dates' set of 24 estimates. For comparison's sake, we superimpose this curve on the curves for the hypothetical plaintiff and defense estimates from Figure 2, and we provide a dashed line at the 0% deviation value. The figure shows that the target firm-date medians are centered at this 0% deviations line, indicating a particular type of accuracy of using the median over each target firm-date's set of 24 *k*-NN parameterizations. However, there is substantial variation over the 10,000 target firm-dates, indicating that the target firm-date median is at best a noisily accurate measure; the density appears to have quite fat tails of the distribution. Panel B isolates attention on the kernel density estimate for the median of parameterizations (note the

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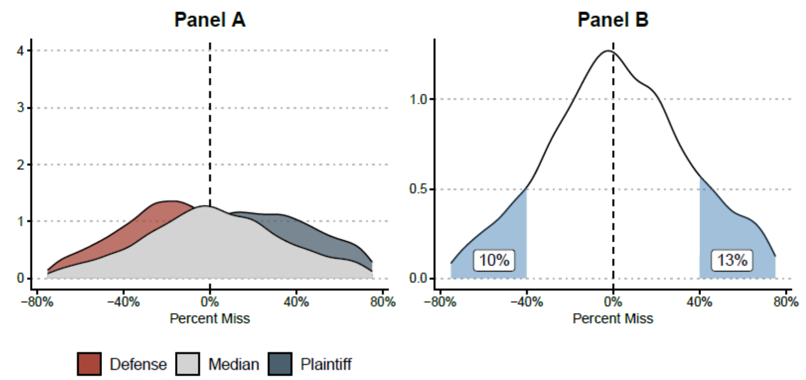
[45] We focus on percentage deviations because not all predictions are on the same scale (scale differences are particularly likely to occur because the median aggregation measure is robust to outliers, whereas the mean is not; consider that the highest valuation ratio among the 5-nearest comps in Table 3 is over 500, whereas the median is 77.55).

[46] In re Appraisal of Dell Inc., No. CV 9322-VCL, 2016 WL 3186538, at *45 (Del. Ch. May 31, 2016), aff'd in part, rev'd in part sub nom. Dell, Inc. v. Magnetar Glob. Event Driven Master Fund Ltd, 177 A.3d 1 (Del. 2017) ("Two highly distinguished scholars of valuation science, applying similar valuation principles, thus generated opinions that differed by 126%, or approximately $28 billion. This is a recurring problem.")

different scales in the two panels). It shows that 10% of the estimates are more than 40% below the true value, and 13% are greater than 40% above.[47]

Figure 3: The Distribution of Permutation Medians



This figure reports the distribution of the median valuation estimate from the permutations of the conventional k-NN approach, measured by the percentage deviation from the true realized market capitalization on the trading date d∗ periods after the end of the fiscal quarter. In Panel A we report this median estimate over the plaintiff and defense density estimates. In Panel B, we report the mass of the distribution falling outside of ±40% of the true market capitalization.

[47] If we consider the full range of estimates, i.e. even those outside of ± 75% of the true value, these numbers are even larger, especially for the distribution's very fat right tail.

B. *Data-Driven Approaches*

As we saw in Section 2, the conventional approach to valuation provides substantial discretion for experts and produces estimates with large variance. In this section, we explore whether modern data-driven approaches to prediction can reduce discretion and variance.[48]

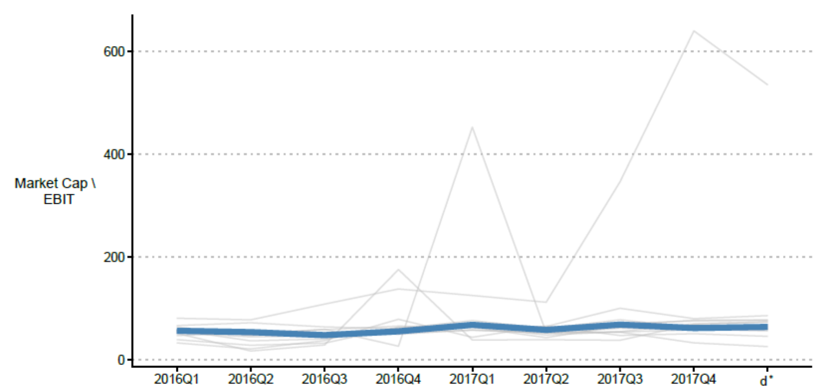
1. Using Time-Series Information

Our first refinement to conventional practice involves using the time series data on valuation ratios, which the conventional *k*-NN approach ignores.

To explain the idea, it will be useful to return to our motivating example involving the target firm Landstar System, with target quarter 2017Q4. Figure 4 plots the time series of Landstar System's valuation ratio, r_{js}, for the 8 fiscal quarters before the target quarter. Landstar System's valuation ratio time series is plotted in blue, and the valuation ratio time series for other firms in the same industry are plotted in gray. Landstar's ratio hovers around 60 throughout this period. Several of the firm's peers have ratios that are similar in magnitude, with a couple of peers exhibiting much larger (and much more variable) ratios. This fact likely explains why the densities from the conventional *k*-NN approach in Figure 1 have such fat right tails (especially when focusing on the mean of the valuation ratios).

[48] It should be noted that both the conventional and data-driven approaches to valuation do not model the control premium. This is consistent with valuation as conducted in appraisal actions under DGCL §262, though it may not be as appropriate for other areas of litigation that use valuation. However, to the extent that the control premium needs to be modeled separately based on the economics of the firm or industry that is subject to the dispute, it can be added to more a more precisely-estimated walk-away value.

Figure 4: Time Series Dynamics of Valuation Ratio



This figure reports the quarterly valuation ratio (market capitalization over EBIT) measures for Landstar and its peer comparables. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

The valuation ratio time series properties in Figure 4 suggest a straightforward method for predicting Landstar's target-date valuation ratio: rather than using the kinds of covariates that textbooks suggest for generating comps, we can use the pre-target-quarter time series of valuation ratios for Landstar and its peers to predict its target-date value. This idea is very much in the spirit of the synthetic controls approach to estimating treatment effects,[49] although we implement the idea using penalized regression.[50]

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[49] See Alberto Abadie et al., Synthetic Control Methods for Comparative Case Studies: Estimating the Effect of California's Tobacco Control Program, 105 J. Am. Stat. Assc. 493-505 (2010) and Nikolay Doudchenko & Guido W. Imbens Balancing, Regression, Difference-in-Differences and Synthetic Control Methods: A Synthesis (Nat'l Bureau of Econ. Research, Working Paper No. 22791, 2016).

[50] Formally, lasso, ridge, and elastic net all can be understood by considering the estimator $\hat{\beta}$ that solves the problem

Table 4: Penalized Regression Weights on Peer Firms

Company	Lasso	Ridge	Elastic Net
INTERCEPT	-14.9506	40.7792	-21.8043
C H ROBINSON WORLDWIDE INC	0.1201	0.0847	0.3367
COVENANT LOGISTICS GROUP INC	0.0000	0.0022	0.0000
FORWARD AIR CORP	0.0000	0.0497	0.0000
HEARTLAND EXPRESS INC	-0.0150	0.0009	-0.0090
HUNT (JB) TRANSPRT SVCS INC	0.7620	0.0479	0.5195
MARTEN TRANSPORT LTD	0.0000	0.0216	0.0000
OLD DOMINION FREIGHT	0.0000	0.0252	0.0000
P.A.M. TRANSPORTATION SVCS	-0.0013	-0.0017	-0.0097
SAIA INC	0.0000	0.0200	0.0000
UNIVERSAL LOGISTICS HLDGS	0.0038	0.0033	0.0495
WERNER ENTERPRISES INC	0.3529	0.0589	0.4605

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the ratio of market capitalization to EBIT for the target firm, and the features that enter the regression are the ratios for the peer firms. We use quarterly data for the preceding two years in fitting the model, and optimize the tuning parameter using leave-one-out cross validation.

Table 4 reports the estimated coefficients from several such models, with potential peer firms' quarterly valuation ratios being the regressors and Landstar's quarterly valuation ratio being the left hand side variable. We use eight quarterly observations in each of

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$$\hat{\beta} = \arg\min_{\beta} \sum_t (y_t - X_t'\beta)^2 + \lambda\left(\alpha \sum_h |\beta_h| + \left(\frac{1-\alpha}{2}\right)\sum_h \beta_h^2\right),$$

for some generic dependent variable y where t indexes observations and h indexes the covariates in X_t and the coefficients β_h that multiply them. The scalars α and λ are tuning parameters. When $\alpha = 0$, $\hat{\beta}$ is the ridge estimator, for which the penalization term is proportional to the sum of squared β-coefficient values but does not depend on the sum of β-coefficient absolute-values. When $\alpha = 1$ instead, $\hat{\beta}$ is the lasso estimator, which does depend on the sum of β-coefficient absolute-values, but does not depend on the sum of squared β-coefficient values. The parameter λ governs the extent to which penalization matters; if $\lambda = 0$, then $\hat{\beta}$ is the ordinary least squares estimator.

the three specifications.[51] The lasso specification yields coefficients of 0 for five of the industry peers, with nonzero coefficients resulting for six of them. Interestingly, two of these are negative, though small, which indicates that the model predicts Landstar's valuation ratio moves in the opposite direction of these companies' ratios, conditional on the other companies'. The ridge specification yields no coefficients with estimated value 0, but it shrinks the coefficients towards each other, so that no industry peer receives a dominant weight. Elastic net produces estimates between lasso and ridge, as one would expect given that its objective function is essentially a combination of the two.[52]

Figure 5 plots the time series of Landstar's valuation ratio in red. The penalized regression models' predicted valuations also appear in the figure. For quarter s, a given model's prediction is computed by (i) predicting the target-quarter valuation ratio (call this $\hat{r}_{jq}$), and (ii) multiplying it by $EBIT_{jq}$. Thus, the penalized regression models' predicted valuations are found as $\hat{v}_{jq} = EBIT_{jq} \times \sum_i r_{is}\hat{\beta}_i$, where i indexes the industry peer firm and $\hat{\beta}_i$ is the model's estimated value of the coefficient on peer firm i's valuation ratio.[53] The fitted ridge estimates (in purple) shrink the fitted value toward the time-series average, smoothing away some of the time series variation in Landstar's true valuation ratio. This smoothing is a hallmark of what penalized regression models are designed to do. The lasso (in green) and elastic net (in blue) estimates are extremely close to each other, and also to the true valuation level (again, in red). The three models produce similar target-date predictions; all are within 4.5% of the true valuation ratio on date d^*.

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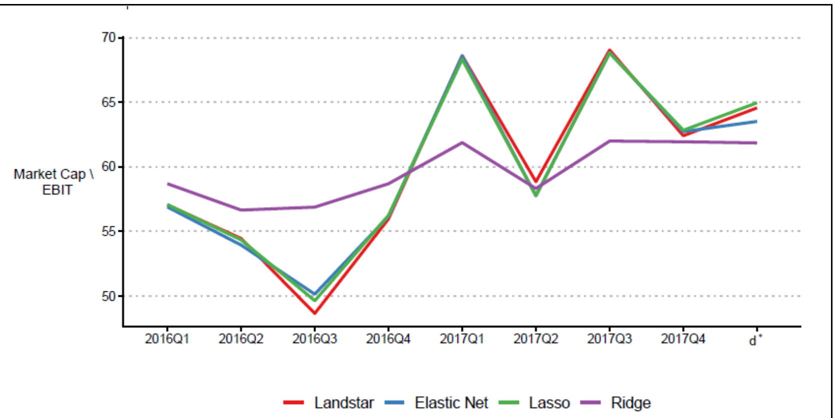
[51] We use leave-one-out cross validation to choose optimal values of the penalized-regression tuning parameter(s) in each specification.

[52] The optimal value for the elastic net specification's parameterization α for Landstar was 0.5.

[53] Note that $\sum_i r_{is}\hat{\beta}_i$ is $\hat{r}_{jq}$, the model prediction of r_{jq}.

Figure 5: Valuation Ratio and Penalized Regression Predictions

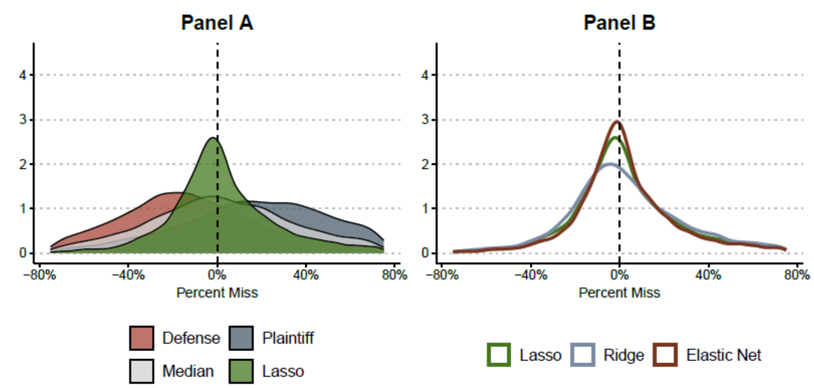


This figure reports the quarterly valuation ratio (market capitalization over EBIT) for Landstar, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d∗ represents the randomly chosen valuation date.

The penalized regression predictions for Landstar are much closer to the true target-date value than were most of the *k*-NN estimates plotted in Figure 1. Of course, it's impossible to determine which of these approaches is superior based on a single example, so we will consider results for our full set of 10,000 simulated target firm-dates.

In Panel A of Figure 6, we superimpose the density estimate of the valuation ratio predictions from the lasso specification over the plaintiff, defense, and median curves discussed above. Even with only eight quarterly observations, the lasso model performs visibly better than the simple median of the 24 *k*-NN permutations. Like the kernel density for the median *k*-NN permutation, the lasso kernel density is centered near zero. However, the lasso density has noticeably lower spread. In Panel B we plot the density estimates for all three penalized regression models (omitting the *k*-NN densities). All three approaches outperform the conventional k-NN approach. The elastic net model's density is a bit more compressed around the 0%-deviation line than the lasso density, which in turn performs better than ridge.

Figure 6: Kernel Density Estimates Using Penalized Regression



This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d∗ periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates over the plaintiff, defense, and median density estimates. In Panel B, we report the three different penalized models separately.

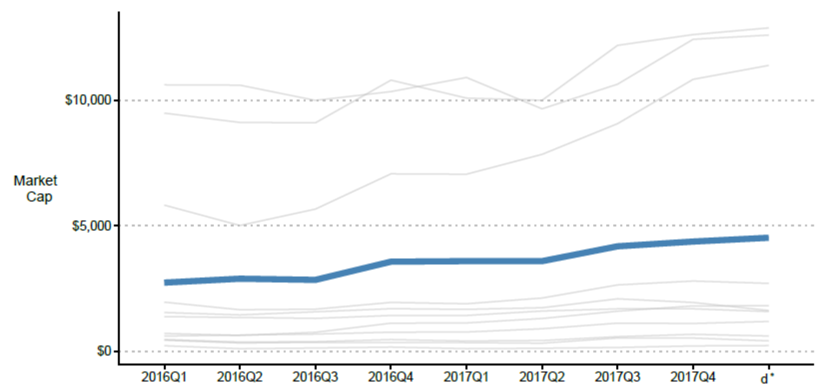
2. Targeting Market Cap Directly

So far, all our prediction approaches have predicted valuation by first predicting the valuation ratio for the target firm-date or firm-quarter, and then multiplying the predicted valuation by the target firm's EBIT for the target quarter. Our next question is whether we can generate further performance improvements by cutting out the middle-man—the valuation ratio—and targeting market cap directly. It makes intuitive sense that targeting market cap directly could improve prediction performance, because the goal of a valuation prediction is market cap itself; the valuation ratio's role is entirely instrumental, and it is impossible that trying to predict it introduces additional noise.

To be sure, it makes sense to target the valuation ratio when using traditional valuations methods because firms that might serve as useful comps may have even if they have very different *levels* of market cap. Targeting the valuation ratio and then "unscaling" by multiplying the predicted valuation ratio by EBIT may be a smart way to use *k*-NN prediction. However, the penalized regression approach can be used in such a way as to reduce or eliminate scale issues (*e.g.*, by including an intercept term, as we did in the models reported in Table 4 above, and by standardizing variables before model fitting). Our next set of estimates therefore will use the quarterly market cap variable, rather than the valuation ratio, as the dependent variable in penalized regression models.

Figure 7 plots the quarterly market capitalization value for Landstar and its industry peers for the two-year (8-quarter) period before the target date. Landstar's market cap trends upward over this period, reflecting growing firm value, as does market cap for many of its peers. However, market caps appear to be relatively less variable around their trends than were the same firms' valuation ratios.

Figure 7: Time Series Dynamics of Market Capitalization



This figure reports the quarterly market capitalization for Landstar and its peer firms. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d∗ represents the randomly chosen valuation date.

Table 5 reports the coefficient estimates from our three penalized regression models, estimated using Landstar's quarterly market cap, rather than its quarterly valuation ratio, as the dependent variable.[54] Eight quarters' worth of lagged market cap are the right hand side variables in these specifications.

Table 5: Penalized Regression Weights on Peer Firms (Market Cap)

Company	Lasso	Ridge	Elastic Net
INTERCEPT	457.9128	1757.9519	383.0429
C H ROBINSON WORLDWIDE INC	0.0232	0.0211	0.0271
COVENANT LOGISTICS GROUP INC	-0.4725	0.1813	-0.5667
FORWARD AIR CORP	0.0000	0.2022	0.0000
HEARTLAND EXPRESS INC	0.0000	0.1494	0.0000
HUNT (JB) TRANSPRT SVCS INC	0.2217	0.0242	0.2304
MARTEN TRANSPORT LTD	0.0000	0.1657	0.0000
OLD DOMINION FREIGHT	0.0000	0.0171	0.0000
P.A.M. TRANSPORTATION SVCS	-2.0200	0.0114	-2.0381
SAIA INC	0.8374	0.0822	0.8225
UNIVERSAL LOGISTICS HLDGS	0.0000	0.2363	0.0000
WERNER ENTERPRISES INC	0.0000	0.0679	0.0000

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is market capitalization for the target firm, and the features that enter the regression are the market capitalizations of the peer firms. We use quarterly data for the preceding two years in fitting the model, and optimize the tuning parameter using leave-one-out cross validation.

Figure 8 replicates the analysis of Figure 7, plotting true quarterly market capitalization for Landstar (in red) alongside the model-fitted values for each of the three penalized regression models.[55] The ridge regression estimates (in purple) again shrink the fitted market cap towards the sample average, while the lasso and elastic net models again produce estimates that closely match the observed valuation over the sample period. In addition, the estimates

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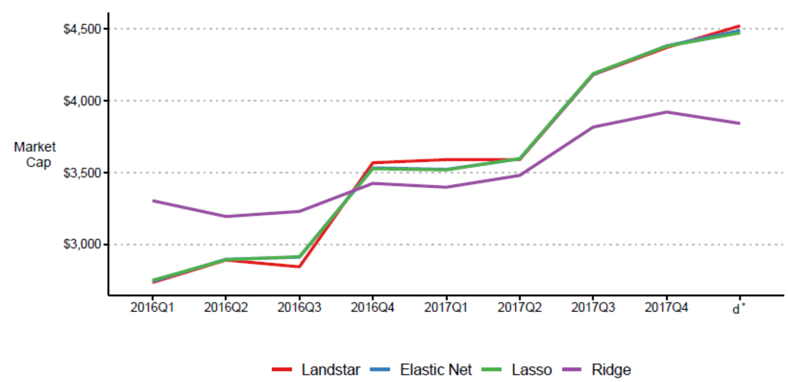
[54] We note that the lasso and elastic net coefficient estimates are quite similar. This reflects the fact that the estimated optimal value of the parameter α in this sample is 0.8, very close to the lasso value of 1.

[55] For a given model, the model-based value of Landstar's market cap in quarter s is $\widehat{mc}_{js} = \sum_i mc_{is}\hat{\beta}_i$. Note that because the elastic net model produces very similar estimates to the lasso model, their lines are largely coextensive.

on the valuation date are much closer for the lasso and elastic net estimates.

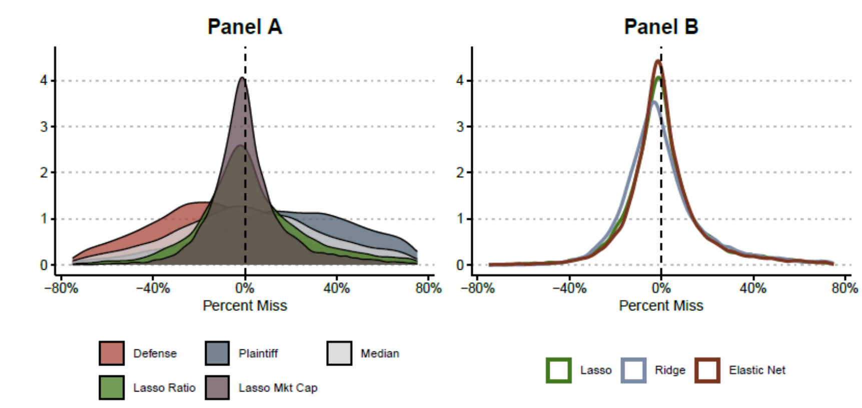
Figure 8: Market Capitalization and Penalized Regression Predictions



This figure reports the market capitalization for Landstar, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

To assess how targeting market cap directly performs in our set of 10,000 simulated target firm-dates, consider Figure 9. In Panel A, we add a kernel density for the lasso-market cap approach to the densities previously plotted in Figure 6. The lasso-market cap density is centered near 0% deviations and has substantially reduced spread; it appears to deliver substantial improvements over the lasso-valuation ratio approach, which Figure 6 showed had itself improved on the k-NN median approach. Panel B plots the market cap-targeting kernel densities for all three penalized regression specifications. Although the three are relatively close, the elastic net model (i.e. choosing an optimal α value to minimize leave-one-out prediction error) produces the best estimates, followed by lasso and ridge.

Figure 9: Kernel Density Estimates Using Penalized Regression



This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d∗ periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates using both the ratio and market capitalization as the outcome variable over the plaintiff, defense, and median density estimates. In Panel B, we report the three different penalized models with market capitalization as the outcome measure separately.

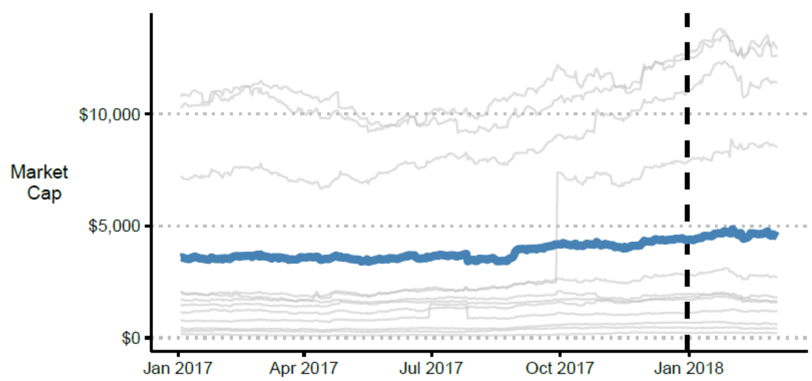
3. Targeting Market Cap and Using Daily Data

We've seen that targeting market capitalization directly, rather than targeting the valuation ratio and then scaling by EBIT, leads to substantial gains in the precision of firm value predictions. Market cap data, unlike EBIT, can be obtained on a daily basis. Because there are many days in a quarter, a natural question is whether we can improve performance yet more by using daily market cap data to predict target-date firm value. Because penalized regression tends to perform better with more granular data, it seems natural that moving to higher-frequency data can further increase predictive performance.

Figure 10 plots the daily market capitalization values for Landstar and its industry peers for the one-year period preceding the randomly-chosen fiscal year end up to the valuation date. Figure 10 is simply the daily-measured corollary to Figure 7. We see again that Landstar's valuation trends upward over the period, and also that

Landstar's market cap exceeds that of most of its peers, with comparatively stable valuation paths.

Figure 10: Daily Time Series Dynamics of Market Capitalization



This figure reports the daily market capitalization for Landstar and its peer comparables. The time series for Landstar is represented in blue while the peers are represented in gray. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter.

Table 6 reports the penalized regression model intercept and coefficient estimates, using Landstar's market cap as the dependent variable and industry peers' daily market caps as right hand side variables. The estimation period spans the 250-day trading period that ends on the date the target quarter ends (i.e., December 31, 2017, for Landstar; note that this means that the target date, March 5, 2018, is considerably outside the estimation period).[56] These estimates are notably less sparse than those that used quarterly data, i.e., all but one industry peer gets a non-zero estimated coefficient. In addition, the estimates are similar across the three specifications (although the elastic net estimates again are closer to the lasso than to the ridge estimates).

[56] Given the longer sample period in these regressions, we use 25-fold cross-validation, rather than more time-intensive leave-one-out optimization, to estimate the penalization parameters.

Table 6: Penalized Regression Weights on Peer Firms (Daily Market Cap)

Company	Lasso	Ridge	Elastic Net
INTERCEPT	190.0837	438.0332	179.5381
C H ROBINSON WORLDWIDE INC	0.1230	0.0884	0.1254
COVENANT LOGISTICS GROUP INC	0.8335	0.5914	0.8870
FORWARD AIR CORP	0.6189	0.4711	0.6252
HEARTLAND EXPRESS INC	0.3143	0.3445	0.3167
HUNT (JB) TRANSPRT SVCS INC	-0.0207	0.0097	-0.0243
KNIGHT-SWIFT TRPTN HLDGS INC	-0.0124	0.0060	-0.0125
MARTEN TRANSPORT LTD	0.1434	0.1089	0.1494
OLD DOMINION FREIGHT	-0.0048	0.0104	-0.0078
P.A.M. TRANSPORTATION SVCS	-0.8385	-0.1693	-0.7733
SAIA INC	0.0000	0.0692	0.0000
UNIVERSAL LOGISTICS HLDGS	1.4166	0.5359	1.3550
WERNER ENTERPRISES INC	0.0392	0.0960	0.0530

This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is daily market capitalization for the target firm, and the features that enter the regression are the market capitalizations of the peer firms. We use daily data starting 250 trading days before fiscal quarter end in fitting the model, and optimize the tuning parameter using 25-fold cross validation.

Figure 11 uses Table 6's coefficient estimates to predict Landstar's market capitalization over the estimation period and for the valuation date. All three penalized regression models (in green for lasso, orange for ridge, and blue for elastic net) predict Landstar's valuation (in red) almost perfectly over the estimation period, although the model-based valuations separate from the true valuation after the estimation period ends on December 31, 2017, (as one would expect). We also investigate the performance of a commonly used but less parametric approach than lasso/ridge/elastic net, namely random forest prediction.[57] The random forest predictions also match the daily valuation of Landstar closely over the estimation period, although it appears to do out of sample than the penalized regression models, which is a sign of possible overfitting in the random forest predictions.

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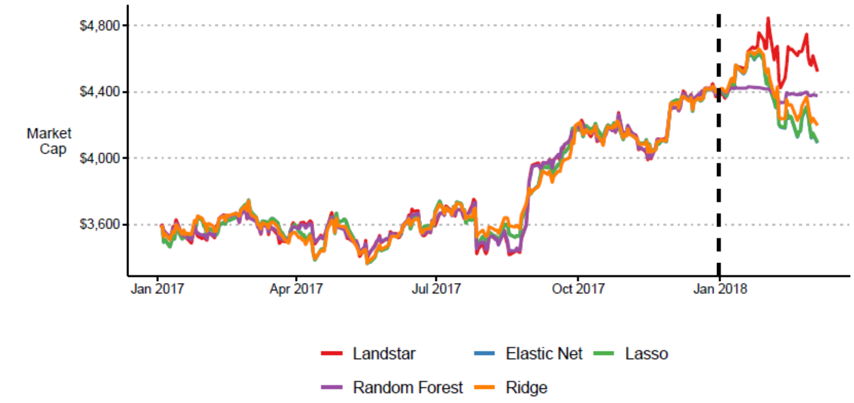
[57] *See* Leo Breiman, *Random Forests*, 45 Machine Learning 5 (2001).

Figure 11: Daily Market Capitalization and Model Predictions

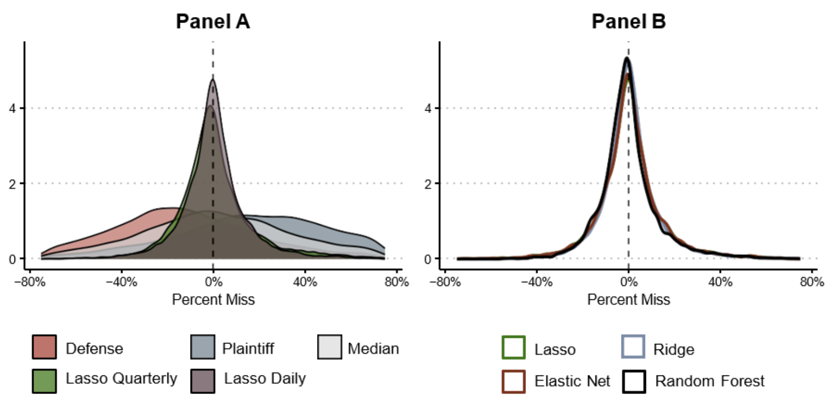


This figure reports the daily market capitalization for Landstar, as well as its predicted value using different penalized regression models. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter, and d^* represents the randomly chosen valuation date.

Turning to our full set of 10,000 target firm-dates, Figure 12's Panel A superimposes a kernel density estimate for the daily market capitalization lasso specification on the set of densities plotted in Figure 9. Daily data further improves on the performance of the quarterly market cap valuation approach, although the improvement seems less dramatic than some of the earlier ones. Panel B separately plots the density for the random forest and the penalized regression prediction approaches. The four perform generally similarly, although the random forest and ridge regression models are best, and roughly equivalent.

Figure 12: Kernel Density Estimates Using Daily Data Penalized Regression

This figure reports the kernel density estimates for the conventional approach as well as for the penalized regression models. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates using both daily and quarterly market capitalization as the outcome variable over the plaintiff, defense, and median density estimates. In Panel B, we report the three different penalized models, as well as the random forest estimates, with market capitalization as the outcome measure.



4. Targeting Daily Returns for Prediction

Once we have switched from quarterly to daily data, there is no obvious reason why we should continue to target the firm's market cap. Because market cap is highly right skewed (i.e., there are some extreme outlier firms with very high market cap), the empirical finance literature almost exclusively focuses on stock price returns rather than market cap.[58] In fact, experts in the other primary area of litigation that uses stock prices—securities fraud litigation—have consistently used returns-based modeling to measure effects related to events at issue in litigation. A natural question is whether litigation involving disputed firm valuations could benefit from taking the same approach.

[58] *See* Stanley J. Kon, *Models of Stock Returns—A Comparison*, 39 J. Fin. 147 (1984).

Let y_{it} represent the stock price return for firm i on day t. We will consider five models, each of whose dependent variable is y_{jt}, where j indexes the target firm (e.g., Landstar in our motivating example). For right hand side variables, we include the daily market-level return (MKTRF) as well as the daily returns from three "factors": the returns on long-short portfolios sorted by size (SMB) and value (HML)[59] and the return on such portfolios sorted by momentum (UMD)[60]. Baker and Gelbach (2020) show that the including peer firms' returns—either via an equally-weighted average or by controlling for the peers separately—can increase the predictive power of event study models, so we include their indexes as well.[61]

We consider five different models using returns as the outcome variables.

1. **FFC-Index**. We model the target firm's daily return as

$$\begin{aligned} y_{jt} \quad = \quad & \alpha + MKTRF_t\beta_1 + SMB_t\beta_2 + HML_t\beta_3 + \\ & UMD_t\beta_4 + PEERINDEX_t\beta_5 + \epsilon_{jt}, \end{aligned}$$

where $E[\epsilon_{jt} | \cdot] = 0$ (with conditioning done on all the covariates) and $PEERINDEX_t$ is the day-t value of the return on the equally-weighted average of the target firm's industry peers.

2. **FFC-All peers**. We model the target firm's daily return as

$$\begin{aligned} y_{jt} \quad = \quad & \alpha + MKTRF_t\theta_1 + SMB_t\theta_2 + HML_t\theta_3 + \\ & UMD_t\theta_4 + \sum_i p_{it}\theta_{5i} + u_{jt}, \end{aligned}$$

[59] These were introduced in Eugene F. Fama and Kenneth R. French, *Size and Book-to-Market Factors in Earnings and Returns*, 50 J. Fin. 131 (1995).

[60] The UMD variable was proposed by Mark M. Carhart, *On Persistence in Mutual Fund Performance*, 52 J. Fin. 57 (1997).

[61] Andrew Baker and Jonah B. Gelbach, Machine Learning and Predicted Returns for Event Studies in Securities Litigation, 5 J. L. Fin. Acct. 231 (2020).

where $E\left[\epsilon_{jt} \middle| \cdot\right] = 0$ (with conditioning again done on all the included covariates). Note that if $\theta_{5i} = \theta_5$, i.e., is constant over all industry peer firms i, then all θ coefficients will equal their numbered β counterparts from the FFC-Index specification.

3. **FFC-Lasso**. This approach involves lasso estimation with the target-firm's daily stock price returns, $\{y_{jt}\}_{t=1}^{T}$, on the left hand side and the set of right hand side variables including an intercept, the three Fama-French-Carhart factors, and the individual peer firm returns; penalization is done in the usual lasso way, by adding the term $\lambda \sum_h |\beta_h|$ to the usual least-squares objective function, where λ is a scalar tuning parameter and h indexes all the covariates included as explanatory variables.

4. **FFC-Ridge**. This approach involves ridge estimation with the same covariates as the lasso specification; the lasso penalization term is replaced by $\lambda \sum_h \beta_h^2$.

5. **FFC-Elastic net**. This approach is essentially a mixture of the lasso and ridge specifications, with the penalization term being $\lambda\left(\alpha \sum_h |\beta_h| + (1-\alpha)\frac{1}{2}\sum_h \beta_h^2\right)$.

All specifications use daily data for the 250 days ending on the last date of the target quarter. We estimate the FFC-Index and FFC-All peers specifications using ordinary least squares. The last three are estimated using standard algorithms, as discussed above.[62] Table 7 reports the coefficient estimates from these five returns-based models for Landstar. The FFC-All peer coefficients have some variation across peer firms, which indicates that the FFC-Index specification foregoes some predictive information by forcing all firms to have the same coefficient. The lasso coefficients include zeros for the value and momentum factors (HML and UMD) as well as for a number of the industry peer firms. However, the estimated elastic net value of α was 0, which is why the elastic net and ridge estimates are identical.

[62] As with the daily market cap specifications, we optimize the tuning parameterizations for the penalized regression models using 25-fold cross validation.

Table 7: Return Coefficients

Company	FFC Index	FFC All Peers	Lasso	Ridge	Elastic Net
INTERCEPT	-0.0003	-0.0003	-0.0002	-0.0002	-0.0002
MKTRF	0.3768	0.1736	0.1667	0.1909	0.1909
SMB	0.1873	0.2107	0.1268	0.1527	0.1527
HML	-0.0088	-0.0224	0.0000	0.0042	0.0042
UMD	0.1063	0.0628	0.0000	0.0789	0.0789
PEER INDEX	0.5993				
C H ROBINSON WORLDWIDE INC		0.1075	0.0716	0.0833	0.0833
COVENANT LOGISTICS GROUP INC		0.0022	0.0000	0.0127	0.0127
FORWARD AIR CORP		0.0975	0.0930	0.0885	0.0885
HEARTLAND EXPRESS INC		0.0681	0.0525	0.0601	0.0601
HUNT (JB) TRANSPRT SVCS INC		0.1095	0.1010	0.0992	0.0992
KNIGHT-SWIFT TRPTN HLDGS INC		-0.0034	0.0000	0.0202	0.0202
MARTEN TRANSPORT LTD		-0.0257	0.0000	0.0224	0.0224
OLD DOMINION FREIGHT		0.2501	0.2641	0.1338	0.1338
P.A.M. TRANSPORTATION SVCS		0.0093	0.0000	0.0099	0.0099
SAIA INC		0.0443	0.0378	0.0593	0.0593
UNIVERSAL LOGISTICS HLDGS		0.0309	0.0175	0.0264	0.0264
WERNER ENTERPRISES INC		0.1133	0.1021	0.0711	0.0711

This table reports the coefficient values on the Fama-French-Carhart factors, and the peer firms, using both ordinary least squares and different forms of penalized regression. The outcome variable is the return on CVS's stock, and the features that enter the regression are the Fama-French-Carhart factors and the returns for the peer firms. We use daily data for the 250 days prior to and ending on the fiscal-year end in fitting the model. For the penalized regression models we optimize the tuning parameter using 25-fold cross validation.

To predict valuation on the target date for each specification in Table 7, we use the same approach as above. Let h index all the explanatory variables in a specification, let X_{iht} be firm i's value of the h^{th} covariate on date t,[63] and let $\hat{\beta}_h$ be the estimated coefficient on X_{iht}. Then the predicted value of the target firm's daily stock price return on any date t is $\hat{y}_{jt} = \sum_i \sum_h X_{iht}\hat{\beta}_h$. Because our objective is to predict the target firm's market cap on date d^*, we need a way to link daily returns to market cap. The target firm's market cap on date t is the product of the number of shares outstanding, S_j,[64] and market price, P_{jt}, so the target firm's market cap on date t is $m_{jt} = S_j \times P_{jt}$. Because y_{jt} is the date-t target-firm daily return, the target firm's market price on date $t+1$ is given by $P_{jt+1} = (1 + y_{jt+1})P_{jt}$.[65] Multiplying by shares outstanding then

[63] For the FFC factors, X_{iht} will be the same for all peer firms on a given date.

[64] We assume that the number of shares outstanding does not change between the end of the estimation period (i.e., the last date of the quarter preceding day d^*) and d^*, which is why S_j does not have a time index, t.

[65] For example, if the firm has a return of $+1\%$ on $t+1$, then $y_{jt+1} = 0.01$, and the firm's price on $t+1$ equals $1.01 \times P_{jt}$.

implies that market cap on $t+1$ is given by $m_{jt+1} = S_j P_{jt+1} = S_j\left(1+y_{jt+1}\right)P_{jt}$. Using the same argument again, market cap on $t+2$ can be found by multiplying m_{jt+1} by $\left(1+y_{jt+2}\right)$. In general, market cap on $d^* > q$, where q is the last date of the target fiscal quarter (the quarter that precedes the target date d^*), is given by

$$\begin{aligned} m_{jd^*} &= S_j P_{jq} \prod_{d=1}^{d^*}\left(1+y_{jq+d}\right) \\ &= m_{jq} CR_j(q, d^*), \end{aligned}.$$

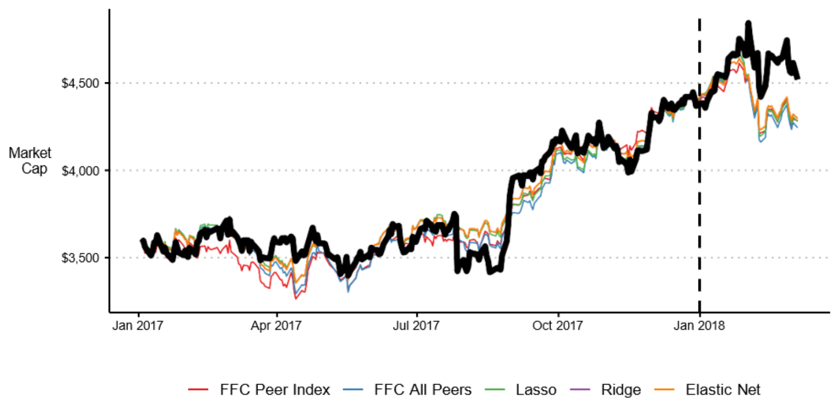
where $m_{jq} = S_j P_{jq}$ is the target firm's market cap on the last day of the target quarter and the term $CR_j(q, d^*) = \prod_{d=1}^{d^*}\left(1+y_{jq+d}\right)$ is the target firm's cumulative return over the period between date q and date d^*.[66]

We report Landstar's true market capitalization (black line) and our models' predictions (various colored lines) in Figure 13. The models generally, if imperfectly, capture the trend in Landstar's market cap over this time period.

[66] In practice, this formula for the cumulative return has to be adjusted to account for any dividends paid during the (q, d^*) window. We do this appropriately in our empirical work but for expositional simplicity, we do not address the issue further in the text.

Figure 13: Daily Market Capitalization and Returns Model Predictions

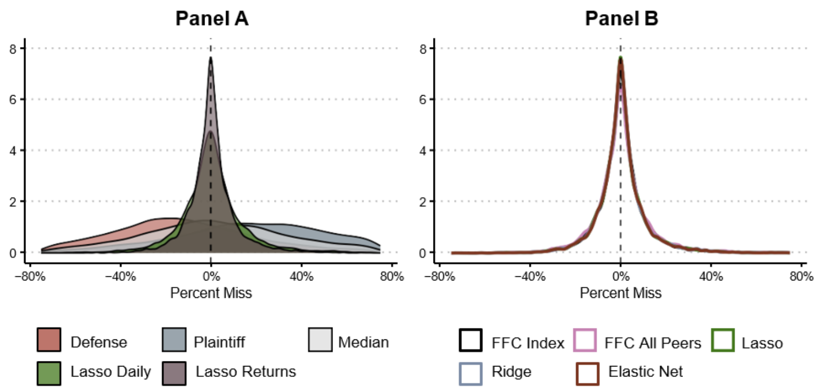
This figure reports the daily market capitalization for Landstar and the prediction from a series of models using returns rather than valuation as the outcome variable. The time series for Landstar is represented in black while the model predictions are in colors. The vertical dashed line represents the end of quarter-end for the randomly sampled quarter.



Finally, we plot kernel density estimates for the deviations of the predicted market capitalization values for our 10,000 target firm-dates in Figure 14. Panel A once again plots the density for the conventional *k*-NN estimates and for the lasso model predicting daily market capitalization, now adding a kernel density for FFC-lasso predictions. The FFC-lasso prediction density is noticeably superior to all the other densities. Interestingly, Panel B shows that the five densities involving daily returns are relatively similar-looking. Notably, *all* of them outperform even the Lasso-daily market cap density displayed in Panel A. Thus it seems that using cumulated predicted daily returns can yield improved valuation predictions even when standard OLS estimation is used with a simple, equal-weighted industry peer index. These findings indicate that it is the use of daily returns that yields the greatest improvement in predictive accuracy relative to other methods.

Figure 14: Kernel Density Estimates Using Daily Data

This figure reports the kernel density estimates for the conventional approach as well as for the returns- based models. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d^* periods after the end of the fiscal quarter. In Panel A we report the lasso regression model estimates using both daily market capitalization and daily returns as the outcome variable over the plaintiff, defense, and median density estimates. In Panel B, we report the different returns-based model estimates separately.



V. A REAL-WORLD APPLICATION: *DFC GLOBAL*

This section applies our approach to the landmark Delaware case of *DFC Global Corp. v. Muirfield Value Partners, L.P.*,[67] a shareholder dispute that metastasized into a famously focal flashpoint for valuation methodology. *DFC Global* marks something of a watershed moment in stockholder appraisal cases, which are statutorily authorized actions brought by dissenting stockholders after the close of certain eligible transactions.[68] By statutory command, the appraisal inquiry focuses on assessing the "fair value" of the target as a going concern, using "all relevant factors," and specifically excluding the value of merger synergies or

[67] 172 A.3d 346 (Del. 2017).

[68] See Choi & Talley (2018).

takeover premiums.[69] The opposing experts in *DFC* used CC as part of their valuation analyses, and the court had to grapple with their divergent opinions.[70]

A. Background

DFC Global[71] is a publicly traded payday lending firm. It faced significant headwinds in 2012-13, including issues related to its financial leverage and regulatory scrutiny in several countries. DFC engaged financial advisor Houlihan-Lockey to advise on a potential sale, and to initiate a bidding process. The bidding process was tumultuous, buffeted by several negative shocks and disappointing earnings reports, which impelled several bidders to withdraw. After contacting over 45 potential buyers, Houlihan eventually corralled 2-3 serious contenders, including a private equity company named Lone Star Funds ("Lone Star"). Ultimately, DFC signed a cash deal with Lone Star at $9.50 per share on April 1, 2014, closing on June 13, 2014. Several DFC stockholders perfected their appraisal rights (led by hedge fund Muirfield Value Partners), and the case landed in front of Chancellor Bouchard of the Delaware Court of Chancery to determine fair value as of the closing date.[72]

B. Expert Opinions

Consistent with longstanding patterns in appraisal litigation under DGCL § 262, much of the substantive analysis in *DFC Global* came down to a valuation dance-off between opposing experts. Kevin Dages of Compass Lexecon, the petitioner's expert, performed a Discounted Cash Flow (DCF) valuation and offered an estimated fair value of $17.90 per share. He also conducted a

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[69] DGCL § 262.

[70] It merits observing that CC and DCF approaches both continue to be part of the standard valuation canon in appraisal proceedings. *See, e.g,* HBK Master Fund v. Pivotal, C.A. No. 20200165-KSJM (Del. Ch. 2023) (ascribing equal weight to DCF and CC analyses).

[71] Ticker: DLLR; CIK: 0001271625; PERMNO: 1627099; PERMCO 46104.

[72] *In Re Appraisal of DFC Global*, 2016 WL 3753123, Jul 08, 2016, at 12.

Comparable Companies analysis, choosing to peg DFC's EBITDA multiple at the 75th percentile among a set of 10 comparable companies that he had identified. Dages reported a wide valuation interval for his CC analysis—between $11.38 and $26.95. Ultimately, Dages relied entirely on his DCF estimate, giving no weight in his final opinion to the CC method, even though a detailed CC analysis was included in his report. In rationalizing this decision, Dages asserted that "[t]he reliability of a multiples-based valuation is highly dependent on the ability to identify sufficiently comparable companies and transactions, or to properly adjust financial performance data to remove non-comparable items."[73]

The respondent offered an expert report from Daniel Beaulne from Duff & Phelps. Like Dages, Beaulne conducted both a DCF and a CC analysis, which yielded estimates of $7.81 per share and $8.07 per share. Unlike Dages, Beaulne provided exclusively point estimates for his valuation approaches,[74] and he gave equal weight to each of the DCF and CC estimates, so that his ultimate valuation opinion was their average, $7.94 per share.

Thus, the per-share valuations offered by Beaulne and Dages were wildly different, with Dages valuing DFC at more than twice Baulne's valuation. This is not an unusual situation in valuation cases.

C. *The Court of Chancery Opinion*

Chancellor Bouchard delivered a 68-page opinion in July 2016, two years after the deal closed. The Chancellor analyzed both experts' DCF and CC analyses, as well as the deal price itself, ultimately drawing from all three channels. As to CC, Bouchard sided with Beaulne's $8.07 figure, largely because Dages was not able to justify his 75th percentile assumption, which he had never deployed in prior valuation reports.[75] In contrast, Chancellor

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[73] Dages Report at 68.

[74] An analysis of Beaulne's expert report suggests that this decision was based in part on the fact that at least one of his earnings multiples rendered a *negative* equity value; rather than excluding it (which would have pushed the valuation higher), he instead averaged this negative multiple with the others. See Beaulne Report at 67-68

[75] Bouchard opinion, 2016 WL 3753123, at 56-57

Bouchard substantially embraced Dages' DCF analysis, adapting it somewhat to deliver his own DCF estimate of $13.07 per share. Chancellor Bouchard's opinion ultimately accorded equal weights to each of the three valuation lenses, with one-third weight apiece: DCF ($13.07/share), Comparable Companies ($8.07/share), and Deal Price ($9.50/share). The end result was a blended assessment of $10.21 per share, which handed a modest victory to the petitioners. Following Lone Star's post-hearing motion, which pointed out an error in the Court's DCF working capital projections, Chancellor Bouchard corrected his math in a revised opinion, but he simultaneously adjusted the perpetuity growth rate assumption as well, resulting in a post-correction valuation that came in at virtually the same figure as the original.[76]

D. *Delaware Supreme Court's Decision*

DFC appealed, making several forceful arguments. The most attention-grabbing of them was the contention that the Chancery Court's discretion should be limited in any appraisal-eligible acquisition that features an arm's-length sale following a competitive bidding process. In such situations, the appellants argued, the transaction price (less synergies) should be the definitive measure of "fair value" under the appraisal statute. This issue alone elicited significant attention, including dueling amicus briefs—one submitted by several law professors, and another submitted by a combined group of legal, economics and finance scholars (even including a Nobel laureate).[77] In the end, the Supreme Court substantially rejected DFC's categorical argument, emphasizing the criticality of preserving the Chancery Court's discretion.[78] On the issue of post-hoc adjustments in the DCF perpetuity growth rate, the Supreme Court held that such changes were not supported by the

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[76] The revised opinion was $10.30, slightly higher than the original. See 172 A.3d at 362.

[77] See Reynolds Holding, *DFC Global Appraisal Battle Draws Opposing Briefs From Professors* (Columbia Blue Sky Blog, Feb. 7, 2017).

[78] *DFC Global Corporation v. Muirfield Value Partners, L.P.* 172 A.3d 346 (Del. 2017).

factual record. The Court remanded the case back to Chancellor Bouchard for reconsideration, after which the case settled.

The key takeaway is that the Supreme Court's *DFC* opinion clearly reinforced the importance of discretion and the need for a detailed explanation by the fact finder of the preferred valuation method. In some cases, a single metric may be most reliable, while in others multiple measuring perspectives should be considered. Perhaps more than anything, however, the case underscores the complexity and discretion involved in corporate valuation. That highlights the value of developing a more principled approach to determining fair value. Even if such an approach does not manage to avoid gaping valuation chasms between competing expert opinions, it can provide guidelines for judges considering which expert approaches are worthy of consideration.

E. *A* DFC *Do-Over?*

This section considers how DFC Global's valuation would have come out using our approaches to CC valuation. We treat as the peer firms of DFC the union of the potential peer firms identified in the Dages and Beaulne expert reports, as well as the other financial firms in the same three-digit SIC code industry.[79] Because the experts' valuations that we discussed above are reported in price per share terms, we take that same approach in this section.[80]

In Figure 15 we report the daily market capitalization for DFC (blue line) and its peers (gray lines) for the two-year period ending on December 31, 2013, which we use as the last "unaffected" market price preceding the appraisal action. For visualization purposes, we report the market capitalization values as percent deviations from the company's market cap as of April 1, 2012. As

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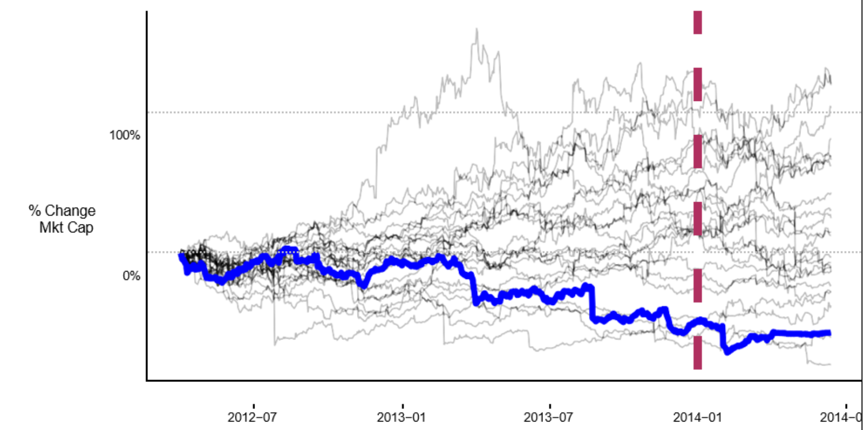
[79] The firm's three digit SIC industry as of fiscal-year 2013 was 609–Functions Related to Depository Banking. We chose not to use the two-digit SIC industry definition, because 60 (Depository Institutions) covered over 500 unique firms at that time.

[80] With a fixed number of shares outstanding during the litigation-relevant period, market cap on any relevant date is just the product of this fixed number and the share price on that date. Thus our results are equally meaningful whether we discuss predicted market cap or predicted share price.

is evident in the data, DFC substantially under-performed most, but not all, of its industry peers over this period.

Figure 15: Daily Market Capitalization Over Time for DFC and Peers

This figure reports the percentage change in market capitalization for DFC and th identified peer firms for the two year period prior to calendar year end 2013, throug the valuation date. The data is measured daily.



A first question is what the predicted valuation would be if we used the penalized regression approach to valuation using only quarterly data on market capitalization, as we did in generating the coefficient estimates reported in section IV.B.1's Table 4; Table 8 reports the resulting lasso, ridge, and elastic net coefficient estimates for DFC Global.

Table 8: Penalized Regression Weights on Peer Firms

Company	Lasso	Ridge	Elastic Net
INTERCEPT	1043.1819	962.8659	1092.2567
ALBEMARLE CORP	0.0032	0.0101	0.0344
BLOCK H & R INC	-0.0412	-0.0046	-0.0145
CASH AMERICA INTL INC	0.0000	-0.0030	-0.0194
CASH CONVERTERS INTERNATIONAL LTD	0.0000	-0.0261	0.0000
CREDIT ACCEPTANCE CORP	0.0000	-0.0151	-0.0156
EURONET WORLDWIDE INC	0.0000	-0.0117	-0.0127
EZCORP INC -CL A	0.0063	0.0202	0.0172
FIRST CASH FINANCIAL SVCS	0.0000	-0.0285	-0.0526
GLOBAL CASH ACCESS HOLDINGS	0.0000	0.0037	0.0000
GLOBAL PAYMENTS INC	0.0000	-0.0093	-0.0016
GREEN DOT CORP	0.0000	-0.0171	-0.0336
HEARTLAND PAYMENT SYSTEMS	0.0000	-0.0247	-0.0405
HIGHER ONE HOLDINGS INC	0.1147	0.0408	0.1127
INTERNATIONAL PERSONAL FINANCE PLC	0.0000	-0.0131	-0.0123
MASTERCARD INC	0.0000	-0.0004	-0.0004
MONEYGRAM INTERNATIONAL INC	0.0000	-0.0343	-0.0913
PROVIDENT FINANCIAL PLC	-0.0331	-0.0159	-0.0248
QC HOLDINGS INC	0.0000	0.4198	0.2083
REGIONAL MANAGEMENT CORP	-0.4393	-0.0753	-0.1400
VERIFONE SYSTEMS INC	0.0000	0.0050	0.0105
VISA INC	0.0000	-0.0004	-0.0005
WESTERN UNION CO	0.0000	0.0012	0.0000
WORLD ACCEPTANCE CORP/DE	0.0000	-0.0624	-0.1289

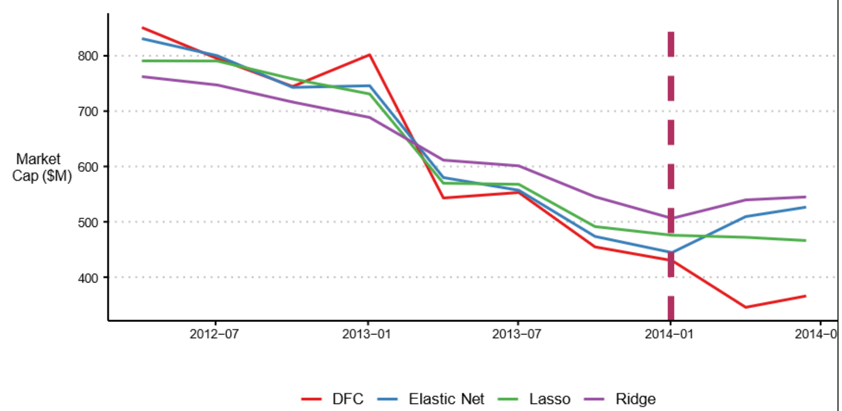
This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the market capitalization for the target firm, and the features that enter the regression are the market capitalization levels for the peer firms. We use quarterly data for the preceding eight calendar quarters in fitting the model, and optimize the tuning parameter using leave-one-out cross validation.

Figure 16 plots the actual market valuation of DFC Global (red line) alongside the model-based predictions of market capitalization generated by the coefficient estimates in Table 8.[81] The penalized models generally do a good job at tracking DFC's valuation over this period. All three models report a valuation above the true realized valuation for DFC (the red line) on the target date of June 13, 2014.

[81] For reference in comparing to the Dages and Beaulne expert reports actually reported in the case, on a per-share basis, the target-date predicted valuations in the figure range from $13.59 to $15.08 per share.

Figure 16: Quarterly DFC Market Capitalization and Model Predictions

This figure reports the quarterly market capitalization for DFC, as well as it predicted value using different penalized regression models. The vertical dashed line represents the end of calendar year 2013.



As noted in section IV.B.2, statistical learning models tend to do better when estimated with larger numbers of observations. That provides reason to believe that an approach that uses daily market capitalization will provide better estimates of valuation than collapsing the value to the quarterly level.[82] Thus, Table 9 reports coefficient estimates from penalized regression models using data on daily rather than quarterly market capitalization, just as we did with the motivating example of Landstar in Table 6 of section IV.B.3. The sample used to estimate the coefficients reported in Table 9 includes daily observations on market capitalization value for the one-year period ending on December 31, 2013, and as above we optimize the tuning parameters using 25-fold cross-validation.

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[82] To be sure, daily observations on market cap also may have a greater signal-to-noise ratio than quarterly ones. We thus do not claim there is any guarantee that daily data will improve performance; this is an empirical question, though one our section IV.B.3 results indicate is answered in favor of preferring daily market cap data to quarterly.

Table 9: Penalized Regression Weights on DFC Global's Peer Firms

Company	Lasso	Ridge	Elastic Net
INTERCEPT	-358.5352	-43.0852	-369.1398
ALBEMARLE CORP	-0.0359	-0.0113	-0.0371
BLOCK H & R INC	-0.0019	-0.0054	-0.0027
CASH AMERICA INTL INC	0.1477	0.1027	0.1477
CASH CONVERTERS INTERNATIONAL LTD	0.0348	0.0186	0.0294
CREDIT ACCEPTANCE CORP	0.1247	0.1022	0.1211
EURONET WORLDWIDE INC	0.1477	-0.0112	0.1156
EZCORP INC -CL A	0.6155	0.2940	0.5830
FIRST CASH FINANCIAL SVCS	-0.2394	-0.1102	-0.2245
GLOBAL CASH ACCESS HOLDINGS	0.4311	0.2914	0.4675
GLOBAL PAYMENTS INC	0.0357	0.0349	0.0438
GREEN DOT CORP	-0.2701	-0.1267	-0.2613
HEARTLAND PAYMENT SYSTEMS	-0.0715	0.0047	-0.0693
HIGHER ONE HOLDINGS INC	0.7660	0.4936	0.7517
INTERNATIONAL PERSONAL FINANCE PLC	-0.0855	-0.0133	-0.0796
MASTERCARD INC	0.0052	-0.0005	0.0041
MONEYGRAM INTERNATIONAL INC	0.0182	0.0458	0.0356
PROVIDENT FINANCIAL PLC	0.0504	-0.0024	0.0486
QC HOLDINGS INC	-1.3996	1.3964	-1.1332
REGIONAL MANAGEMENT CORP	-0.3071	-0.0531	-0.1938
VERIFONE SYSTEMS INC	-0.0369	0.0028	-0.0330
VISA INC	-0.0084	-0.0037	-0.0077
WESTERN UNION CO	0.0290	0.0161	0.0281
WORLD ACCEPTANCE CORP/DE	0.1417	-0.0198	0.1261

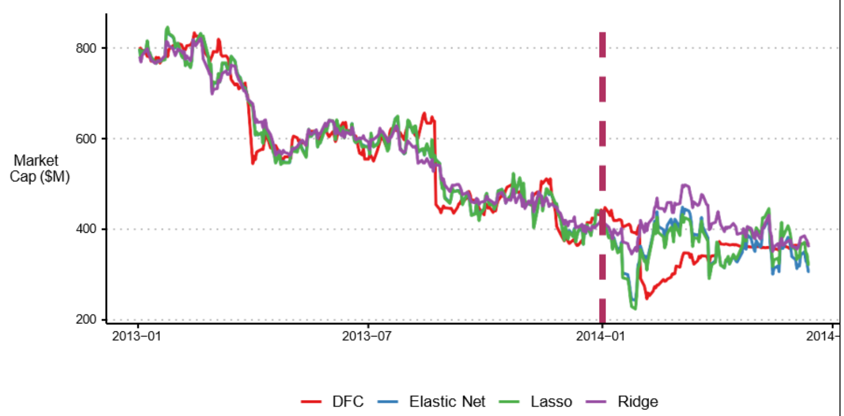
This table reports the coefficient values on the peer firms using different forms of penalized regression. The outcome variable is the market capitalization for the target firm, and the features that enter the regression are the market capitalization levels for the peer firms. We use daily data for the one-year period ending in 2013, and optimize the tuning parameter using 25-fold cross-validation.

Figure 17 shows how the model predictions associated with these coefficient estimates compare to the realized value for DFC over the estimation period and the valuation date. As above, the penalized regression models' predicted market cap largely track the actual market cap series over the estimation period. Moreover, the target-date predictions are quite close to DFC Global's actual market capitalization after announcement.[83]

[83] The implied price per share of these predicted valuations is $6.98 to $9.74, depending on the model.

Figure 17: Daily DFC Market Capitalization and Model Predictions

This figure reports the daily market capitalization for DFC, as well as its predicte value using different penalized regression models. The vertical dashed lin represents the end of calendar year 2013.



Finally, we consider using daily stock price returns in place of daily market cap, just as we did in section IV.B.4. We report the coefficient estimates from returns-based penalized regressions in Table 10. Most potential peer firms have coefficients equal to 0 in the lasso specification, and the same is true for the elastic net estimates, which are very close to the lasso estimates. Thus, data-driven estimation rejects using most of DFC Global's industry peers.

Table 10: Return Coefficients

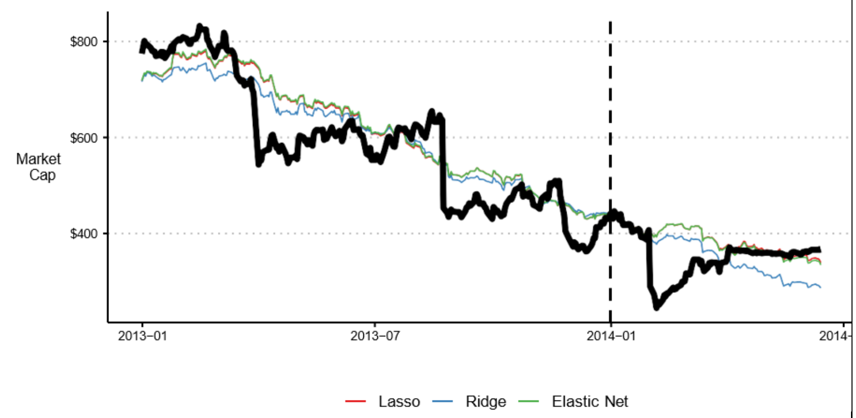
Company	Lasso	Ridge	Elastic Net
INTERCEPT	-0.0022	-0.0025	-0.0022
MKTRF	0.4596	0.2016	0.4523
SMB	0.1199	0.2871	0.1594
HML	0.0000	0.2679	0.0000
UMD	0.0000	0.0384	0.0000
ALBEMARLE CORP	0.0000	0.0603	0.0000
BLOCK H & R INC	0.0655	0.0881	0.0763
CASH AMERICA INTL INC	0.3231	0.1662	0.3259
CASH CONVERTERS INTERNATIONAL LTD	0.0000	-0.0233	0.0000
CREDIT ACCEPTANCE CORP	0.0000	0.0006	0.0000
EURONET WORLDWIDE INC	0.0000	0.0292	0.0000
EZCORP INC -CL A	0.0000	0.0428	0.0000
FIRST CASH FINANCIAL SVCS	0.0000	0.0139	0.0000
GLOBAL CASH ACCESS HOLDINGS	0.0000	-0.0105	0.0000
GLOBAL PAYMENTS INC	0.0000	0.0208	0.0000
GREEN DOT CORP	0.0000	-0.0048	0.0000
HEARTLAND PAYMENT SYSTEMS	0.0000	0.0397	0.0000
HIGHER ONE HOLDINGS INC	0.0408	0.0560	0.0486
INTERNATIONAL PERSONAL FINANCE PLC	0.0000	-0.0184	0.0000
MASTERCARD INC	0.0000	0.0265	0.0000
MONEYGRAM INTERNATIONAL INC	0.0000	0.0296	0.0000
PROVIDENT FINANCIAL PLC	0.0000	0.0264	0.0000
QC HOLDINGS INC	0.0000	-0.0253	0.0000
REGIONAL MANAGEMENT CORP	0.0000	0.0311	0.0000
VERIFONE SYSTEMS INC	0.0000	-0.0176	0.0000
VISA INC	0.0000	0.0431	0.0000
WESTERN UNION CO	0.0000	0.0469	0.0000
WORLD ACCEPTANCE CORP/DE	0.0952	0.1002	0.1038

This table reports the coefficient values on the Fama-French-Carhart factors, and the peer firms, using different forms of penalized regression. The outcome variable is the return on DFC's stock, and the features that enter the regression are the Fama-French-Carhart factors and the returns for the peer firms. We use daily data for the year prior to and ending on December 31, 2013, in fitting the model. We optimize the tuning parameter using 25-fold cross validation.

Figure 18 plots the actual market cap for DFC Global (black line), together with the predicted values from the daily stock price returns penalized regression models. As noted, the lasso and elastic net coefficient estimates are virtually identical, so the time series of their predicted valuations are visually indistinguishable. The results are largely consistent across penalization type, with the predicted DFC Global value on the target date equivalent to approximately \$8.04 to \$9.19 per share.

Figure 18: DFC Daily Market Capitalization and Returns Model Predictions

This figure reports the daily market capitalization for DFC and the prediction from a series models using returns rather than valuation as the outcome variable. The time series for DFC represented in black while the model predictions are in colors. The vertical dashed line represen the end of calendar year 2013.



We offer Table 11 to summarize the various DFC Global valuations we discussed above. The first two blocks of rows in the table report the actual case's experts' valuations, which are roughly $8 for Beaulne (the defense's expert) and a range of roughly $11 to $27 for Dages (the plaintiff's expert). The bottom block reports our various predicted valuations using (i) quarterly market cap data, discussed in section IV.B.2, (ii) daily market cap data, discussed in section IV.B.3, and (iii) daily stock price return data, discussed in section IV.B.4.[84]

Our predicted valuations range from a low of roughly $8 using daily market cap data to roughly $13 using quarterly market cap data. Our simulation results show that the daily market cap data approach outperforms the quarterly market cap data approach. However, our results also show that the daily stock price return approach substantially outperforms every other approach, including the daily market cap approach. The daily stock price return approach

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[84] The "weighted-average" results in this table weight each of the three penalized regression model predictions by the inverse of the cross-validation squared-error. This is a sensible weighting approach because lower cross-validation error is associated with greater estimation precision.

yields a predicted DFC Global target-date valuation of roughly $10 per share. If we were picking among these predicted valuations, we would pick the daily stock price returns approach, so our preferred valuation would be in the neighborhood of $10. Coincidentally, Chancellor Bouchard arrived at essentially the same place.

Table 11: Fair Value Estimates – DFC

Source	Type	Estimate
Petitioner's Expert		
Dages		
	DCF	$17.90
	CC	$11.38 - $26.95
Respondent's Expert		
Beaulne		
	DCF	$7.81
	CC	$8.07
	Overall	$7.94
Data-Driven Approach		
Quarterly Market Cap		
	Lasso	$12.08
	Ridge	$14.11
	Elastic Net	$13.65
	Weighted Avg	$13.25
Daily Market Cap		
	Lasso	$8.32
	Ridge	$9.41
	Elastic Net	$7.93
	Weighted Avg	$8.58
Daily Returns		
	Lasso	$10.56
	Ridge	$9.24
	Elastic Net	$10.44
	Weighted Avg	$10.08

VI. LIMITATIONS AND EXTENSIONS

The simulation results in Section 4 show that our data-driven approach offers substantial benefits for the practice of financial valuation in litigation. This is especially true when we use daily data.

That is straightforward when the targeted value is a firm's market capitalization, but it creates a challenge if the dispute centers around a different measure of value, such as the target firm's enterprise value. The problem is that enterprise value depends not just on market cap, but also on debt, and firms generally report aggregate debt levels only on a quarterly basis, via SEC filings (quarterly, with Form 10-Q, and annually with Form 10-K).

Ultimately, this should not be an insurmountable challenge for litigation valuation. Under bankruptcy priority rules, equity gets paid out only after debt; consequently, enterprise value calculations typically are based on the book value of debt. Thus, we can use the penalized regression approach with daily data to value equity's residual claim, and then carry-forward the most recent quarterly measure of the book value of debt. Critically, this means there's no need to value firm debt on a daily basis.[85]

Another potential area of improvement in data-driven valuation is model averaging. In each case we discussed above, there are many potential valuation estimates, depending on the choice of model and data. In other settings, model-averaging has proven quite effective at capturing the inherent uncertainty in prediction exercises where the true underlying data-generating process is unknown.[86] To assess this idea, we take the predicted values using daily market cap data. There are three penalized regression predictions for each date, as well as one random forest prediction. For each of our 10,000 Monte Carlo simulations we calculate the model averaged estimate as:

$$m = \frac{1}{4} \sum_{k} (w_k \times V_k)$$

where k denotes one of the four underlying prediction models (lasso, ridge, elastic net, and random forest), V_k is the predicted valuation for model k on date d^*, and w_k is the inverse of the estimation period cross-validation squared error. Figure 19 plots the density of the

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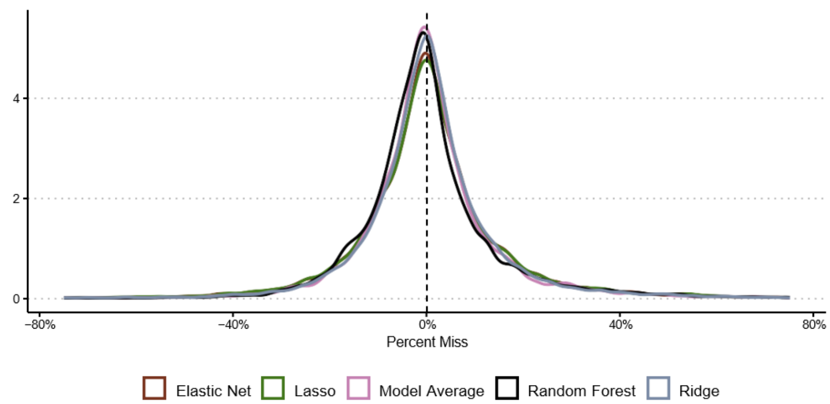
[85] If the firm issued or redeemed an anomalous quantity of debt during the litigation-relevant period, one could value the firm's target-quarter debt level separately using the quarterly penalization approach derived above.

[86] For a discussion of the use of model-averaging in the Netflix competition, see Hal R. Varian, *Beyond Big Data*, 49.1 Bus. Econ. 27 (2014).

underlying model and the model average for our Monte Carlo sample. Not surprisingly given its construction, the model average does well in comparison to the underlying models. However, it doesn't provide visually noticeable predictive improvement over the best-performing individual models.

Figure 19: Model Averaging

This figure reports the kernel density estimates for the machine-learning approaches to valuation using daily market capitalization as the outcome variable, as well as a model-averaged estimate. The reported value is measured by the percentage deviation from the true realized market capitalization on the trading date d^* periods after the end of the fiscal quarter. The model average is the weighted average of the underlying machine learning estimates, with the inverse of the cross-validation squared error as the weight.



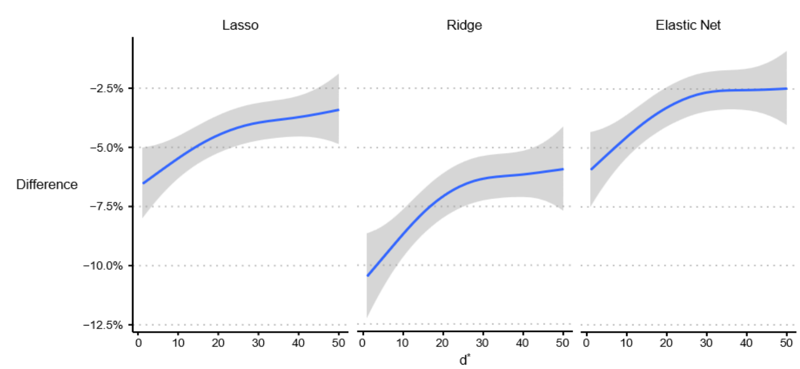
Finally, we note that by construction, our simulation approach partly hamstrings the performance of the models. In all cases, we restricted the end of the estimation period to q, the last date of the target quarter. This is necessary for the *k*-NN approach. It is *un*necessary for our other approaches, because daily data are available on market cap and stock price returns for the period between q and d^*. We ignored that fact in because we wanted to assess the relative performance of the various data-driven approaches when these are restricted to the same data used in *k*-NN prediction.

In practice, though, there will typically be dates between q and d^* that are reasonable to include in the estimation period using daily data. That can be expected to further improve the performance of methods that can use daily data. To assess this claim, Figure 20 plots the performance gains from using daily data rather than quarterly data, as a function of the length of time between quarter end, q, and valuation, d^*, for each model. To calculate performance improvement for each of the 10,000 target firm-dates, we calculate the percentage miss for the daily-data prediction (call this PM_d) and the percentage miss for the quarterly-data prediction (call this PM_q). Define $\Delta = |PM_d| - |PM_q|$; when the daily-data approach misses by more than the quarterly-data approach, regardless of the direction in which the miss occurs, Δ is positive. When instead the quarterly-data approach misses by more, Δ is negative.

Figure 20 shows that Δ is more negative for cases in which the valuation date d^* is closer to the end of the quarter, q. This indicates that more improvement is gained from using daily data when less time elapses between the end of the target quarter and the target date. This pattern suggests that augmenting our above approach by extending the estimation period past q can be expected to yield even more performance gains.

Figure 20: Model Improvement by d^*

This figure reports the absolute difference in the percentage miss between the daily and quarterly measured market capitalization for the penalized regression models (lasso, ridge, and elastic net) by the randomly- selected number of days post quarter-end (d^*) used as the target date. The reported curve is a smoothed generalized additive model of the difference in performance on the number of days post-quarter end, with the estimated relationship reported with a blue line, and the confidence interval for the estimate reported in grey.



VII. CONCLUSION

In this paper, we critically examine the practice of financial valuation in litigation. We show that conventional methods used by experts have two serious drawbacks. First, they accord experts enormous discretion to report valuations that benefit the parties the experts represent, in ways that make it difficult to observe such bias directly. Second, conventional methods far underperform feasible alternatives. We believe the approaches we identified—most notably, predicting firm value using daily stock price returns—should be adopted in valuation litigation wherever possible.

DATA APPENDIX

Abbreviation	Definition	Formula
age	Company Age	quarter_num - first_quarter + 1
assets	Total Assets	atq
capitalization_ratio	Capitalization Ratio	debt / (ev + cheq)
debt	Total Debt	dlttq + dlcq
debt_equity	Debt to Equity Ratio	debt + mkt_cap
ebit_growth	EBIT Growth	ebit / lag(ebit, 4) - 1
ebit_margin	EBIT Margin	ebit / sales
ebitda_growth	EBITDA Growth	ebitda / lag(ebitda, 4) -
ebitda_margin	EBITDA Margin	ebitda / sales
eps_growth	Earnings Per Share Growth	oepsxq / lag(oepsxq, 4)
ev	Enterprise Value	prccq * cshoq + pstkrq dlcq + dlttq + mibq - cl
gross_profit_margin	Gross Profit Margin	(sales - cogsq) / sales
implied_dividend_yield	Implied Dividend Yield	(dvpspq * 4) / prccq
leverage_ratio	Leverage Ratio	debt / ebitda
log_ret	Log Return	log(ret + 1)
mkt_cap	Market Capitalization	prccq * cshoq
mkt_cap_ebit	Market Capitalization to EBIT Ratio	mkt_cap / ebit
ni_growth	Net Income Growth	niq / lag(niq, 4) - 1
ni_margin	Net Income Margin	niq / sales
roa	Return on Assets	niq / avg_assets
roe	Return on Equity	niq / avg_equity
roic	Return on Invested Capital	ebit / avg_debt_equit
sales	Total Sales	saleq

This table present the variables used in the conventional matching approach formula for their construction, as well as the underlying variables as defined i CRSP. In addition, if used as a matching variable then we report which sou variable in their determination set.